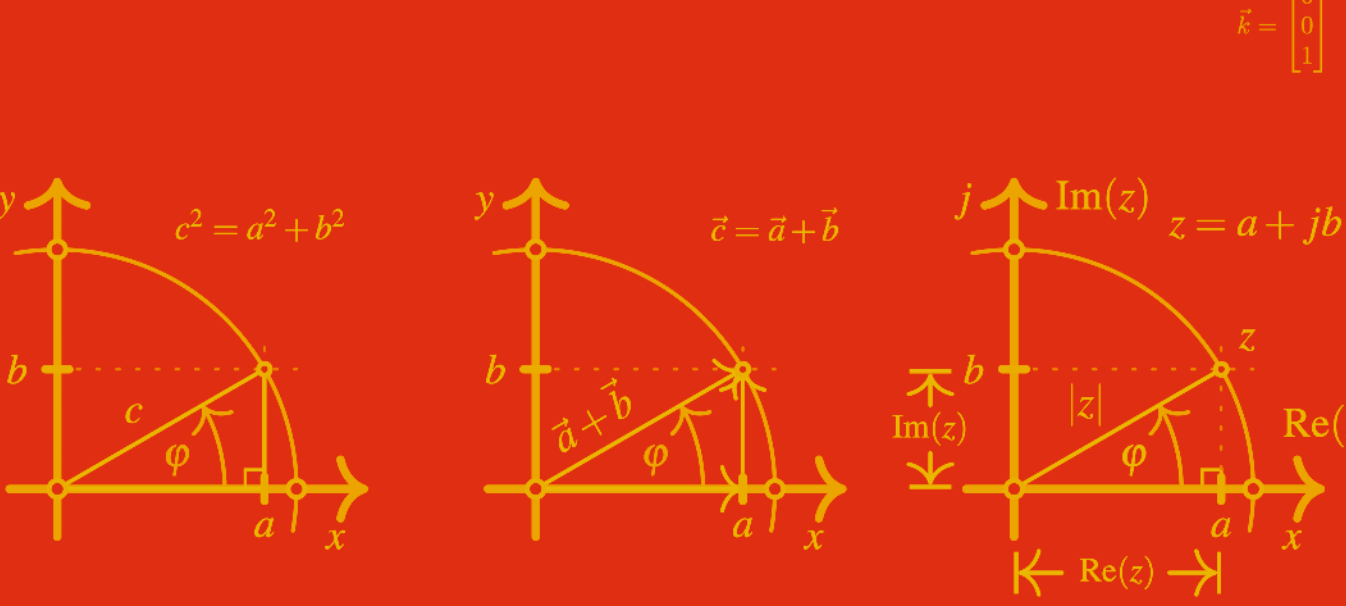


Robert Sobot

Engineering Mathematics by Example

Vol. I: Algebra and Linear Algebra

Second Edition

Springer

Engineering Mathematics by Example

Robert Sobot

Engineering Mathematics by Example

Vol. I: Algebra and Linear Algebra

Second Edition

Springer

Robert Sobot
ETIS-ENSEA
Cergy-Pontoise, France

ISBN 978-3-031-41199-1 ISBN 978-3-031-41200-4 (eBook)
https://doi.org/10.1007/978-3-031-41200-4

1st edition: © Springer Nature Switzerland AG 2021
2nd edition: © The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2023
This work is subject to copyright. All rights are solely and exclusively licensed by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.
The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.
The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Paper in this product is recyclable.

To my math teacher Mr. Miloš Belušević

Preface

Preface to the Second Edition

It is inevitable that first edition of any type of textbook, and especially textbooks for mathematics, includes quite a few errors that slipped by all preproduction reviews. In this second edition, errors are thoroughly reviewed and corrected, with hopes that not many new ones are created.

As the original volume doubled, this second edition is split into three separate books, Vol. I, II, III. In order to reinforce natural development in the study, best effort is put to logically organize the presented examples and techniques so that subsequent problems reference the once already solved.

Île-de-France, France — Robert Sobot
June 27, 2023

Preface to the First Edition

This tutorial book resulted from my lecture notes developed for undergraduate engineering courses in mathematics that I teach over the last several years at l'École Nationale Supérieure de l'Électronique et de ses Applications (ENSEA), Cergy in Val d'Oise department, France.

My main inspiration to write this tutorial type collection of solved problems came from my students who would often ask "How do I solve this? It is impossible to find the solution," while struggling to logically connect all the little steps and techniques that are required to combine together before reaching solution. In the traditional classical school systems mathematics used to be thought with the help of systematically organized volumes of problems that help us develop "the way of thinking." In other words, to learn how to apply the abstract mathematical concepts to everyday engineering problems. Same as for music, it is also true for mathematics that in order to reach high level of competence one must put daily effort into studying of typical forms over long period of time.

In this tutorial book I choose to give not only the complete solutions to the given problems, but also guided hints to techniques being used at the given moment. Therefore, problems presented in this book do not provide review of the rigorous mathematical theory, instead the theoretical background is assumed, while this set of classic problems provides a playground to play and to adopt some of the main problem-solving techniques.

The intended audience of this book are primarily undergraduate students in science and engineering. At the same time, my hope is that students of mathematics at any level will find this book to be useful source of practical problems to practice.

Île-de-France, France Robert Sobot
November 30, 2020

Acknowledgments

I would like to acknowledge all those classic wonderful collections of mathematical problems that I grew up with and used as the source of my knowledge, and I would like to say thank you to their authors for providing me with thousands of problems to work on. Specifically, I would like to acknowledge classic collections written in ex–Yugoslavian, Russian, English and French languages, some are listed in the bibliography. Hence, I do want to acknowledge their contributions that are clearly visible throughout this book, which are now being passed on to my readers.

I would like to thank all of my former and current students who I had opportunity to tutor in mathematics since my student years. Their relentless stream of questions posed with unconstrained curiosity forced me to open my mind, to broaden my horizons, and to improve my own understandings.

I want to acknowledge Mr. Allen Sobot for reviewing and verifying some of the problems, for very useful discussions and suggestions, as well as his work on technical redaction in first edition of this book.

Sincere gratitude goes to my publisher and editors for their support and making this book possible.

Contents

1 Numbers 1

Problems 1
- 1.1 Basic Number Operations 1
- 1.2 Fractional Powers and Radicals 2
- 1.3 Undefined Forms 3
- 1.4 Absolute Numbers and Expressions 3

Answers 4
- 1.1 Basic Number Operations 4
- 1.2 Fractional Powers and Radicals 5
- 1.3 Undefined Forms 7
- 1.4 Absolute Numbers and Expressions 8

2 Polynomials 15

Problems 15
- 2.1 Polynomial Expansion 15
- 2.2 Binomial Theorem (Pascal's Triangle) 16
- 2.3 Long Division 16
- 2.4 Factorization 17
- 2.5 Difference of Squares 18
- 2.6 Quadratic Polynomial—Viète Formulas 19
- 2.7 Completing the Square 20
- 2.8 Factor Theorem 20
- 2.9 Partial Fraction Decomposition 21

Answers 23
- 2.1 Polynomials 23
- 2.2 Binomial Theorem (Pascal's Triangle) 24
- 2.3 Long Division 25
- 2.4 Factorization 27
- 2.5 Difference of Squares 29
- 2.6 Quadratic Polynomial—Viète Formulas 31
- 2.7 Completing the Square 34
- 2.8 Factor Theorem 35
- 2.9 Partial Fraction Decomposition 44

3 Linear Equations and Inequalities 55

Problems 55
3.1 Linear Equations 55
3.2 System of Linear Equations 55
3.3 Linear Inequalities 56
3.4 System of Linear Inequalities 56

Answers 57
3.1 Linear Equations 57
3.2 System of Linear Equations 60
3.3 Linear Inequalities 63
3.4 System of Linear Inequalities 70

4 Logarithmic and Exponential Functions 75

Problems 75
4.1 Logarithmic and Exponential Functions 75
4.2 Simple Logarithmic Calculations 75
4.3 Exponential Equations 76
4.4 Logarithmic Equations 76
4.5 Exponential–Logarithmic Equations 77
4.6 Exponential Inequalities 77
4.7 Logarithmic Inequalities 77

Answers 78
4.1 Logarithmic and Exponential Functions 78
4.2 Simple Logarithmic Calculations 82
4.3 Exponential Equations 84
4.4 Logarithmic Equations 88
4.5 Exponential–Logarithmic Equations 89
4.6 Exponential Inequalities 90
4.7 Logarithmic Inequalities 93

5 Trigonometry 97

Problems 98
5.1 Trigonometric Definitions 98
5.2 Basic Calculations 99
5.3 Basic Identities 100
5.4 Equations 101
5.5 Inequalities 101

Answers 102
5.1 Trigonometric Definitions 102
5.2 Basic Calculations 107

5.3 Basic Identities . . . 112
5.4 Equations . . . 123
5.5 Inequalities . . . 131

6 Complex Algebra . . . 137

Problems . . . 137
6.1 Basic Complex Number Forms . . . 137
6.2 Polar Forms . . . 138
6.3 Complex Plane . . . 138
6.4 Euler Identity . . . 139
6.5 Rational Powers . . . 139
6.6 Complex Equations . . . 140

Answers . . . 141
6.1 Basic Complex Number Forms . . . 141
6.2 Polar Forms . . . 147
6.3 Complex Plane . . . 149
6.4 Euler Identity . . . 152
6.5 Rational Powers . . . 161
6.6 Complex Equations . . . 164

7 Bode Plot . . . 171

Problems . . . 172
7.1 Basic Complex Functions . . . 172
7.2 Bode Plot Examples . . . 173

Answers . . . 174
7.1 Basic Complex Functions . . . 174
7.2 Bode Plot Examples . . . 193

8 Linear Algebra . . . 199

Problems . . . 199
8.1 Vector Definitions . . . 199
8.2 Vector Operations . . . 200
8.3 Linear Transformations . . . 201
8.4 Determinants . . . 202
8.5 Cramer's Rule . . . 203
8.6 Vector Space . . . 204
8.7 Eigenvalues and Eigenvectors . . . 205
8.8 Matrix Inversion . . . 206
8.9 Powers of Diagonalizable Matrices . . . 207

Answers . . . 208
8.1 Vector Definitions . . . 208
8.2 Vector Operations . . . 211
8.3 Linear Transformations . . . 217
8.4 Determinants . . . 222
8.5 Cramer's Rule . . . 228
8.6 Vector Space . . . 246

8.7 Eigenvalues and Eigenvectors 255
8.8 Matrix Inversion 285
8.9 Powers of Diagonalizable Matrices 288

Bibliography 301

Index 303

Acronyms

$f(x)$, $g(x)$	Functions of x
$f'(x)$, $g'(x)$	First derivatives of functions of $f(x)$, $g(x)$
$f(g(x))$	Composite function
$\sin(x)$	Sine of x
$\cos(x)$	Cosine of x
$\tan(x)$	Tangent of x
$\arctan(x)$	Arctangent of x
$\text{sinc}\,(t)$	Sine cardinal of t
$\log(x)$	Logarithm of x, base 10
$\ln(x)$	Natural logarithm of x, base e
$\log_n(x)$	Logarithm of x, base n
$x * y$	Convolution product
i	Imaginary unit, $i^2 = -1$
j	Imaginary unit, $j^2 = -1$
$\Im(z)$	Imaginary part of a complex number z
$\Re(z)$	Real part of a complex number z
$\mathscr{F}(x(t))$	Fourier transform of $x(t)$
$X(\omega)$	Fourier transform of $x(t)$
$\mathscr{L}^{-1}\{F(s)\}$	Inverse Laplace transform of $F(s)$
$\mathscr{F}^{-1}(X(\omega))$	Inverse Fourier transform of $X(\omega)$
$\overset{F}{\rightarrow}$	Apply Fourier transform
$\overset{LT}{\longrightarrow}$	Apply Laplace transform
$\overset{\text{l'H}}{=}$	Apply l'Hôpital rule
DFT	Discrete Fourier transform
FFT	Fast Fourier transform
$\delta(x)$	Dirac distribution
$\Lambda(x)$	Triangular distribution
$\Pi(x)$	Square (rectangular) distribution
Ш_T	Dirac comb whose period is T
$\Gamma(t)$	Heaviside step function of t
$u(t)$	Heaviside step function of t
$r(t)$	Ramp function of t
$\text{sign}\,(t)$	Sign function of t
(a, b)	Pair of numbers
(a, b)	Open interval
$\equiv$	Equivalence

$\{a_1, a_2, a_3, \dots\}$	List of elements, vector, series
$\{a_n\}$	List of elements, vector, series
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{N}$	The set of natural numbers
$\mathbb{Q}$	The set of rational numbers
$a \in \mathbb{C}$	Number a is included in the set of complex numbers
$a \in \mathbb{R}$	Number a is included in the set of real numbers
$\forall x$	For all values of x
$\angle$	Angle, argument
$\Rightarrow$	It follows that
$\frac{dy}{dx}$	Derivative of y relative to x
u'_x	Partial derivative of $u(x, y, z, \dots)$ relative to x
$u''_{x,y}$	Second partial derivative of $u(x, y, z, \dots)$, relative to x then to y
z^*	Complex conjugate of number z
$\|x\|$	Absolute value of x
$A_{m,n}$	Matrix A whose size is $m \times n$
I_n	The identity square matrix whose size is $n \times n$
$\|A\|$	Determinant of matrix A
A^T	Transpose of matrix A
Δ	Difference between two variables
Δ	Main determinant of a matrix
Δ_x	Cramer's sub-determinant relative to variable x
$\vec{a}$	Vector a
$\sum_{i=0}^{\infty} a_i$	Sum of elements $\{a_0, a_1, \dots, a_\infty\}$
dB	Decibel
dBm	Decibel normalized to "10^{-3}", i.e., "milli"
$H(jx)$	Transfer function
$x \to \underset{\twoheadleftarrow}{a}$	Limiting to a from its right side (x-axis)
$x \to \underset{\twoheadrightarrow}{a}$	Limiting to a from its left side (x-axis)
$x \to a\downarrow$	Limiting to a from above (y-axis)
$x \to a\uparrow$	Limiting to a from below (y-axis)
$\{\cdot\}$	in–line hints

Numbers

1

Basic number operations and rules, Assuming $a \in \mathbb{R}$ and $a \neq 0$,

$$a^0 \stackrel{\text{def}}{=} 1, \quad (a \neq 0)$$

$$a^1 \stackrel{\text{def}}{=} a$$

$$a^{-n} \stackrel{\text{def}}{=} \frac{1}{a^n}, \quad (a \neq 0)$$

$$a^{\frac{n}{m}} \stackrel{\text{def}}{=} \sqrt[m]{a^n}, \quad (a \geq 0)$$

$$|f(x)| \stackrel{\text{def}}{=} \begin{cases} f(x) & ; \ \big(f(x) \geq 0\big) \\ -f(x) & ; \ \big(f(x) < 0\big) \end{cases}$$

$$a^{n+1} = a^n \cdot a$$

$$a^n \, a^m = a^{m+n}$$

$$\big(a^n\big)^m = a^{n\,m} = \big(a^m\big)^n$$

$$(a\,b)^n = a^n \, b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} = a^n \, b^{-n}, \quad (b \neq 0)$$

$$\frac{a^n}{a^m} = a^n \, a^{-m} = a^{n-m}, \quad (a \neq 0)$$

Problems

1.1 Basic Number Operations

Calculate expressions in P.1.1 to P.1.7.

1.1. $(-2)^{-3}$

1.2. $\left(\frac{1}{2}\right)^{-2}$

1.3. $\left(\frac{1}{2}\right)^{-2} + \left(\frac{1}{2}\right)^{-1}$

1.4. $\dfrac{(-2)^{-3} - (-3)^{-2}}{(-4)^{-1}} \quad \left(\frac{2}{3}\right)^{-3}$

1.5. $\big((-2)^{-1} + (-3)^{-1}\big) \div \big((-3)^{-1} - (-6)^{-1}\big)$

1.6. $-4 \times 10^4 + 2.5 \times 10^5$

1.7. $0.5^{-1} + 0.25^{-2} + 0.125^{-3} + 0.0625^{-4}$

Calculate expressions in P.1.8 to P.1.15.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023

R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41200-4_1

1.8. 2^0 **1.9.** $\left(\frac{1}{2}\right)^0$ **1.10.** $\left(-\frac{3}{8}\right)^0$

1.11. $(\sqrt{3}+\pi-1)^0$ **1.12.** $(\cos x)^0,\ (x \neq \pm\pi/2)$ **1.13.** $\left(e^3\right)^0$

1.14. $\left(\frac{3}{2}+\frac{\pi}{6}\right)^0-\left(\frac{4}{27}-\frac{7\pi}{9}\right)^0$ **1.15.** $\left(\frac{3}{2}+\frac{\pi}{6}\right)^0-\left(\frac{4}{27}-\frac{2^2}{3^3}\right)^0$

Calculate expressions in P.1.16 to P.1.18.

1.16. $2^{3000}\times 3^{2000}$ **1.17.** $3^{200}\times 4^{-300}$ **1.18.** $5^{-2000}\times 2^{3000}$

1.2 Fractional Powers and Radicals

Calculate expressions in P.1.19 to P.1.23.

1.19. $\left(\frac{16}{25}\right)^{\frac{1}{2}}$ **1.20.** $\left(\frac{1}{27}\right)^{-\frac{4}{3}}$ **1.21.** $\left(\frac{1}{2^{-6}}\right)^{\frac{3}{2}}$

1.22. $\left(\frac{1}{1024}\right)^{-\frac{2}{5}}$ **1.23.** $\left(\frac{1}{4}\right)^{-\frac{3}{2}}+\left(\frac{8}{27}\right)^{\frac{2}{3}}$ **1.24.** $\left(\frac{1}{256}\right)^{0.375}$

1.25. $0.25^{-0.5}$ **1.26.** $25^{-\frac{3}{2}}$ **1.27.** $0.001^{-\frac{1}{3}}$

Calculate expressions in P.1.28 to P.1.38.

1.28. $x^{-\frac{2}{5}}$ **1.29.** $x^{-\frac{3}{20}}y^{-\frac{7}{15}}$ **1.30.** $x^{\frac{1}{8}}y^{\frac{2}{25}}$

1.31. $\dfrac{x^{-\frac{1}{2}}y^{\frac{3}{4}}}{z^{-\frac{7}{8}}v^{\frac{5}{3}}}$ **1.32.** $\dfrac{x^{-\frac{5}{2}}}{y^{\frac{3}{4}}}\ z^{-\frac{7}{12}}$ **1.33.** $\left(x^{\frac{2}{3}}y^{\frac{5}{12}}\right)^{-0.75}$

1.34. $\left(x^n x^{\frac{1}{n-1}}\right)\div\left(x^{n^2}\right)^{\frac{1}{n-1}}$ **1.35.** $x^{\frac{2}{3}}\ x^{\frac{3}{4}}\ x^{-1}$ **1.36.** $x^{\frac{3}{2}}\ y^{\frac{4}{5}}\ z^{\frac{5}{6}}\div\left(x^{\frac{5}{4}}\ y^{\frac{2}{3}}\ z^{\frac{7}{12}}\right)$

1.37. $\left(\left(x^{\frac{3}{4}}\right)^2\right)^{-\frac{1}{3}}\div\left(\left(x^{-1}\right)^{\frac{1}{2}}\right)^{-\frac{2}{3}}$ **1.38.** $\left(\left(x^{\frac{3}{4}}\right)^2\right)^{-\frac{1}{3}}\div\left(\left(x^{-1}\right)^{\frac{1}{2}}\right)^{-\frac{2}{3}}$

Calculate expressions in P.1.39 to P.1.42.

1.39. $\left(\sqrt[3]{4a^2b}\right)^2$

1.40. $\sqrt[a]{x^n} \div \sqrt[n]{x^a}$

1.41. $x\sqrt{x\sqrt{x\sqrt[3]{x}}}\sqrt[3]{x\sqrt{x}}$

1.42. $\sqrt[x]{a^{3x-4}}\ \sqrt[x]{a^{1-x}}\ \sqrt[x]{a^{3-x}}$

1.3 Undefined Forms

Calculate and/or discuss expressions in P.1.43 to P.1.51.

1.43. 1^{∞}

1.44. 0^0

1.45. $0 \cdot \infty$

1.46. ∞^0

1.47. $\infty - \infty$

1.48. $\dfrac{\infty}{\infty}$

1.49. $\dfrac{0}{0}$

1.50. $\dfrac{27(2^3-8)}{1-27\cdot 3^{-3}}$

1.51. $\left((-2)^{-3}+\dfrac{1}{8}\right)\dfrac{1}{64-2^6}$

1.4 Absolute Numbers and Expressions

Calculate expressions in P.1.52 to P.1.58.

1.52. $|x-3| = 7$

1.53. $|3-x| > 4$

1.54. $||x|-5| > 4$

1.55. $\dfrac{|x|}{x}$

1.56. $\dfrac{|x|+x}{2}$

1.57. $\dfrac{|x|+2x}{3}+|x|+x$

1.58. $\dfrac{|x-|2x||}{|x|+x}-\dfrac{|x-2|}{|x|+1}>0$

1.59. $\left(\dfrac{x-2+|x-2|}{2}\right)^2+\left(\dfrac{x-2-|x-2|}{2}\right)^2$

Answers

1.1 Basic Number Operations

1.1. $(-2)^{-3} = (-1 \cdot 2)^{-3} = \{(a\,b)^n = a^n\,b^n\} = (-1)^{-3}(2)^{-3} = (-1)\,\dfrac{1}{2^3} = -\dfrac{1}{8}$

1.2. $\left(\dfrac{1}{2}\right)^{-2} = \left\{a^{-n} = \dfrac{1}{a^n}\right\} = 2^2 = 4$

1.3. $\left(\dfrac{1}{2}\right)^{-2} + \left(\dfrac{1}{2}\right)^{-1} = 2^2 + 2 = 6$

1.4.
$$\frac{(-2)^{-3} - (-3)^{-2}}{(-4)^{-1}} \left(\frac{2}{3}\right)^{-3} = \frac{(-1/2)^3 - (-1/3)^2}{-1/4} \times \left(\frac{3}{2}\right)^3$$
$$= \frac{-1/8 - 1/9}{-1/4} \times \frac{27}{8} = \left(\frac{1}{8} + \frac{1}{9}\right) \times \cancel{4} \times \frac{27}{\cancel{8}\,2}$$
$$= \frac{17}{72} \times \frac{27}{2} = \frac{17 \times 3 \times \cancel{9}}{\cancel{9} \times 8 \times 2} = \frac{51}{16}$$

1.5. Negative exponent identity: $a^{-n} \equiv 1/a^n$, thus

$$\left((-2)^{-1} + (-3)^{-1}\right) \div \left((-3)^{-1} - (-6)^{-1}\right) = \left(-\frac{1}{2} - \frac{1}{3}\right) \div \left(-\frac{1}{3} + \frac{1}{6}\right) = -\frac{5}{6} \div \left(-\frac{1}{6}\right)$$
$$= \frac{5}{\cancel{6}} \times \cancel{6} = 5$$

1.6. Numbers with the equal powers are easier to operate with

$$-4 \times 10^4 + 2.5 \times 10^5 = -4 \times 10^4 + 25 \times 10^4 = 21 \times 10^4 = 2.1 \times 10^5$$

1.7. Decimal numbers have their equivalent fractional forms.

$$0.5^{-1} + 0.25^{-2} + 0.125^{-3} + 0.0625^{-4} = \left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{4}\right)^{-2} + \left(\frac{1}{8}\right)^{-3} + \left(\frac{1}{16}\right)^{-4}$$
$$= 2 + 4^2 + 8^3 + 16^4 = 2 + (2^2)^2 + (2^3)^3 + (2^4)^4 = 2 + 2^4 + 2^9 + 2^{16} = 2 + 16 + 512 + 65{,}536$$
$$= 66{,}066 \quad \text{(note the powers of two)}$$

1.8. By definition, assuming a base not equal zero nor infinity, zero powers equal one, as $2^0 = 1$

1.9. See P.1.8, thus $\left(\dfrac{1}{2}\right)^0 = 1$

1.10. $\left(-\dfrac{3}{8}\right)^0 = 1$

1.11. $(\sqrt{3}+\pi-1)^0 = 1$

1.12. Excluding the case 0^0, i.e.$(x \neq \pm\pi/2)$, then $(\cos x)^0 = 1$

1.13. $\left(e^3\right)^0 = \left\{(a^n)^m \quad = a^{n\,m}\right\} = e^{3\cdot 0} = e^0 = 1$

1.14. $\left(\frac{3}{2}+\frac{\pi}{6}\right)^0 - \left(\frac{4}{27}-\frac{7\pi}{9}\right)^0 = 1-1 = 0$

1.15. $\left(\frac{3}{2}+\frac{\pi}{6}\right)^0 - \left(\frac{4}{27}-\frac{7\pi}{9}\right)^0 = 1-1 = 0$

1.16. Exceedingly large powers may only be simplified, for example reduced to the same base, as

$$2^{3000}\times 3^{2000} = (2^3)^{1000}\times(3^2)^{1000} = 8^{1000}\times 9^{1000} = 72^{1000}$$

1.17. $3^{200}\times 4^{-300} = \frac{3^{200}}{(4^3)^{100}} = \frac{3^{200}}{64^{100}} = \frac{3^{200}}{(8^2)^{100}} = \frac{3^{200}}{8^{200}} = \left(\frac{3}{8}\right)^{200}$

1.18. $5^{-2000}\times 2^{3000} = \frac{(2^3)^{1000}}{(5^2)^{1000}} = \left(\frac{8}{25}\right)^{1000}$

1.2 Fractional Powers and Radicals

Fractional powers are equivalent cardinals, and they obey the same rules as "ordinary" powers.

1.19. $\left(\frac{16}{25}\right)^{\frac{1}{2}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}$

1.20. $\left(\frac{1}{27}\right)^{-\frac{4}{3}} = 27^{\frac{4}{3}} = \left(3^3\right)^{\frac{4}{3}} = 3^{\left(\cancel{3}\times\frac{4}{\cancel{3}}\right)} = 3^4 = 81$

1.21. $\left(\frac{1}{2^{-6}}\right)^{\frac{3}{2}} = \left(2^6\right)^{\frac{3}{2}} = \left\{(a^n)^m \quad = a^{n\,m}\right\} = 2^{\cancel{6}^{3}\times\frac{3}{\cancel{2}}} = 2^9 = 512$

1.22. $\left(\frac{1}{1024}\right)^{-\frac{2}{5}} = 1024^{\frac{2}{5}} = \left(2^{10}\right)^{\frac{2}{5}} = 2^{\cancel{10}^{2}\times\frac{2}{\cancel{5}}} = 2^4 = 16$

1.23. $\left(\frac{1}{4}\right)^{-\frac{3}{2}} + \left(\frac{8}{27}\right)^{\frac{2}{3}} = 4^{\frac{3}{2}} + \left(\frac{2^3}{3^3}\right)^{\frac{2}{3}} = \left(2^2\right)^{\frac{3}{2}} + \left(\frac{2}{3}\right)^{\left(\cancel{3}\times\frac{2}{\cancel{3}}\right)} = 2^{\left(\cancel{2}\times\frac{3}{\cancel{2}}\right)} + \frac{4}{9} = 8+\frac{4}{9}$

$$= \frac{8\times 9+4}{9} = \frac{76}{4} = 19$$

1.24. $\left(\frac{1}{256}\right)^{0.375} = \left(\frac{1}{2^8}\right)^{\frac{3}{8}} = \left(\frac{1}{2}\right)^{\left(\cancel{8}\times\frac{3}{\cancel{8}}\right)} = \frac{1}{8}$

1.25. $0.25^{-0.5} = \left(\frac{1}{4}\right)^{-\frac{1}{2}} = 4^{\frac{1}{2}} = \sqrt{4} = \pm 2$

1.26. $25^{-\frac{3}{2}} = \left(5^2\right)^{-\frac{3}{2}} = 5^{\left(\cancel{2}\times\frac{-3}{\cancel{2}}\right)} = 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

1.27. $0.001^{-\frac{1}{3}} = \left(10^{-3}\right)^{-\frac{1}{3}} = 10^{-\cancel{3}\,1\left(-\frac{1}{\cancel{3}}\right)} = 10^{(-1)(-1)} = 10^1 = 10$

1.28. $x^{-\frac{2}{5}} = \left(x^2\right)^{-\frac{1}{5}} = \left(\frac{1}{x^2}\right)^{\frac{1}{5}} = \frac{1}{\left(x^2\right)^{\frac{1}{5}}} = \frac{1}{\sqrt[5]{x^2}}$

1.29. $x^{-\frac{3}{20}}\,y^{-\frac{7}{15}} = \left(x^3\right)^{-\frac{1}{20}}\left(y^7\right)^{-\frac{1}{15}} = \left(\frac{1}{x^3}\right)^{\frac{1}{20}}\left(\frac{1}{y^7}\right)^{\frac{1}{15}} = \frac{1}{\left(x^3\right)^{\frac{1}{20}}}\,\frac{1}{\left(y^7\right)^{\frac{1}{15}}} = \frac{1}{\sqrt[20]{x^3}\,\sqrt[15]{y^7}}$

1.30. $x^{\frac{1}{8}}\,y^{\frac{2}{25}} = \sqrt[8]{x}\,\sqrt[25]{y^2}$

1.31. $\frac{x^{-\frac{1}{2}}\,y^{\frac{3}{4}}}{z^{-\frac{7}{8}}\,v^{\frac{5}{3}}} = \frac{\frac{1}{\sqrt{x}}\sqrt[4]{y^3}}{\frac{1}{\sqrt[8]{z^7}}\sqrt[3]{v^5}} = \frac{\sqrt[8]{z^7}\,\sqrt[4]{y^3}}{\sqrt{x}\,\sqrt[3]{v^5}}$

1.32. $\frac{x^{-\frac{5}{2}}}{y^{\frac{3}{4}}}\;z^{-\frac{7}{12}} = \frac{1}{x^{\frac{5}{2}}}\,\frac{1}{y^{\frac{3}{4}}}\,\frac{1}{z^{\frac{7}{12}}} = \frac{1}{\sqrt{x^5}\,\sqrt[4]{y^3}\,\sqrt[12]{z^7}}$

1.33. $\left(x^{\frac{2}{3}}\,y^{\frac{5}{12}}\right)^{-0.75} = \left(x^{\frac{2}{3}}\,y^{\frac{5}{12}}\right)^{-\frac{3}{4}} = x^{\left(-\frac{\cancel{2}}{\cancel{3}}\frac{\cancel{3}^1}{\cancel{4}_2}\right)}\,y^{\left(-\frac{5}{\cancel{12}_4}\frac{\cancel{3}}{\cancel{4}}\right)} = x^{-\frac{1}{2}}\,y^{-\frac{5}{16}} = \frac{1}{\sqrt{x}\,\sqrt[16]{y^5}}$

1.34. $\left(x^n\,x^{\frac{1}{n-1}}\right) \div \left(x^{n^2}\right)^{\frac{1}{n-1}} = \left(x^n\,x^{\frac{1}{n-1}}\right)\left(x^{n^2}\right)^{-\frac{1}{n-1}} = \begin{Bmatrix} a^n\,a^m & = a^{m+n} \\ \left(a^n\right)^m & = a^{n\,m} \end{Bmatrix} = x^{n+\frac{1}{n-1}}\,x^{-\frac{n^2}{n-1}}$

$$= x^{\frac{n^2-n+1}{n-1}}\,x^{-\frac{n^2}{n-1}} = \left\{a^n\,a^m = a^{m+n}\right\} = x^{\frac{\cancel{n^2}-n+1-\cancel{n^2}}{n-1}} = x^{\frac{-(n-1)\,1}{\cancel{n-1}}}$$

$$= x^{-1} = \frac{1}{x},\ (n \neq 1)$$

1.35. $x^{\frac{2}{3}}\,x^{\frac{3}{4}}\,x^{-1} = \left\{a^n\,a^m = a^{m+n}\right\} = x^{\frac{2}{3}+\frac{3}{4}-1} = x^{\frac{5}{12}} = \sqrt[12]{x^5}$

1.36. $x^{\frac{3}{2}}\,y^{\frac{4}{5}}\,z^{\frac{5}{6}} \div \left(x^{\frac{5}{4}}\,y^{\frac{2}{3}}\,z^{\frac{7}{12}}\right) = \left\{a^{-n} = \frac{1}{a^n}\right\} = \left\{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b}\,\frac{d}{c}\right\}$

$$= x^{\frac{3}{2}-\frac{5}{4}}\,y^{\frac{4}{5}-\frac{2}{3}}\,z^{\frac{5}{6}-\frac{7}{12}} = x^{\frac{1}{4}}\,y^{\frac{2}{15}}\,z^{\frac{1}{4}} = \sqrt[4]{y}\,\sqrt[15]{y^2}\,\sqrt[4]{z}$$

1.37. $\left(\left(x^{\frac{3}{4}}\right)^2\right)^{-\frac{1}{3}} \div \left(\left(x^{-1}\right)^{\frac{1}{2}}\right)^{-\frac{2}{3}} = \left(x^{\frac{3}{4}}\right)^{-\frac{2}{3}} \left(\left(x^{-1}\right)^{\frac{1}{2}}\right)^{\frac{2}{3}} = x^{-\frac{6}{12}}\, x^{-\frac{2}{6}}$
$= x^{-\frac{1}{2}-\frac{1}{3}} = x^{-\frac{5}{6}} = \dfrac{1}{\sqrt[6]{x^5}}$

1.38. $\left(\left(x^{\frac{3}{4}}\right)^2\right)^{-\frac{1}{3}} \div \left(\left(x^{-1}\right)^{\frac{1}{2}}\right)^{-\frac{2}{3}} = \left\{\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b}\dfrac{d}{c}\right\} = x^{(3/4)\,2)\,2)\,(-1/3)}\; x^{(-1)\,(1/2)\,(2/3)}$
$= x^{-\frac{1}{2}-\frac{1}{3}} = x^{-\frac{5}{6}} = \dfrac{1}{\sqrt[6]{x^5}}$

1.39. $\left(\sqrt[3]{4a^2b}\right)^2 = \sqrt[3]{2^4\, a^4 b^2} = \sqrt[3]{2 \cdot 2^3 a^3 a\, b^2} = \left\{\left(a^m\right)^n = \left(a^n\right)^m\right\} = \sqrt[3]{2(2a)^3 a\, b^2} = 2a\sqrt[3]{2ab^2}$

1.40. $\sqrt[a]{x^n} \div \sqrt[n]{x^a} = x^{\frac{n}{a}} \div x^{\frac{a}{n}} = x^{\frac{n}{a}}\, x^{-\frac{a}{n}} = x^{\frac{n^2-a^2}{an}}$

1.41. Roots are easier to handle in the form of fractional powers, thus starting from most inner root

$$x\sqrt{x\sqrt{x\sqrt[3]{x}}}\,\sqrt[3]{x\sqrt{x}} = x\sqrt{x\sqrt{x\,x^{\frac{1}{3}}}}\,\sqrt[3]{x\,x^{\frac{1}{2}}} = x\sqrt{x\sqrt{x^{\frac{4}{3}}}}\,\sqrt[3]{x^{\frac{3}{2}}} = x\sqrt{x x^{\frac{2}{3}\,\frac{1}{2}}}\; x^{\frac{3}{2}\,\frac{1}{3}}$$
$$= x\sqrt{x^{\frac{5}{3}}}\, x^{\frac{1}{2}} = x\, x^{\frac{5}{3}\,\frac{1}{2}}\, x^{\frac{1}{2}} = x^{1+\frac{5}{6}+\frac{1}{2}} = x^{\frac{14}{6}\,\frac{7}{3}} = x^{\frac{7}{3}} = x^{2+\frac{1}{3}} = x^2\sqrt[3]{x}$$

1.42. Whether specific numbers or variables, identities apply as

$$\sqrt[x]{a^{3x-4}}\,\sqrt[x]{a^{1-x}}\,\sqrt[x]{a^{3-x}} = a^{\frac{3x-4}{x}}\, a^{\frac{1-x}{x}}\, a^{\frac{3-x}{x}} = a^{\frac{3x-4+1-x+3-x}{x}} = a^{\frac{x}{x}} = a, \quad (x \neq 0)$$

1.3 Undefined Forms

1.43. Strictly speaking there is ∞ term in the expression, thus one argument may be that therefore 1^∞ is not defined. However, equally valid argument is that this case is special because number one is the neutral element for multiplication, thus $1^\infty = 1 \cdot 1 \cdot 1 \cdots = 1$. The answer depends on the context. In the higher level mathematical theory, rigorous comparing of infinities is still in the research domain, thus not considered here.

1.44. Any number to the power of zero is assumed equal one, that is to say including $0^0 = 1$. However, depending on the context this expression may be declared undefined.

1.45. Expression $0 \cdot \infty$ is not defined. That is to say, this product may produce any arbitrary number as the result. For example, consider the following two cases:

$$
\begin{array}{rr}
(10)\cdot(10)=100 & (2)\cdot(3)=6 \\
\left(\frac{10}{100}\right)\ (10\cdot 100)=100 & \left(\frac{2}{200}\right)\ (3\cdot 200)=6 \\
\left(\frac{10}{1000}\right)\ (10\cdot 1000)=100 & \left(\frac{2}{2000}\right)\ (3\cdot 2000)=6 \\
(\cdots)\ (\cdots)=100 & (\cdots)\ (\cdots)=6 \\
\left(\frac{10}{\infty}\right)\ (10\cdot\infty)=100 & \left(\frac{2}{\infty}\right)\ (3\cdot\infty)=6 \\
(0)\ (\infty)=100 & (0)\ (\infty)=6
\end{array}
$$

Which is to say that by starting with any product of two numbers, it is possible to show that $0\cdot\infty$ product equal that same product. This is because any non–zero number divided by infinity equal zero, and any non–zero number multiplied by infinity equal infinity.

1.46. The form of '∞^0' is assumed undefined. Through the properties of logarithmic function, this power form may be converted into $0/0$ or $0\cdot\infty$, thus normally declared undefined.

1.47. Difference of two infinite terms '$\infty-\infty$' is undetermined. Comparing multiple infinites is still in the research domain.

1.48. Ratio of two infinite terms '∞/∞' is a classic undefined form. With simple algebra it can be converted into '$0/0$' form.

1.49. Ratio of two zero terms '$0/0$' is a classic undefined form. With simple algebra it can be converted into '∞/∞' form. (Recall that $1/0=\infty$ and $1/\infty=0$).

1.50. $$\frac{27(2^3-8)}{1-27\cdot 3^{-3}}=\frac{27(0)}{1-27/27}=\frac{0}{0}\quad\text{(undefined)}$$

1.51. $$\left((-2)^{-3}+\frac{1}{8}\right)\frac{1}{64-2^6}=\left(-\frac{1}{8}+\frac{1}{8}\right)\frac{1}{0}=0\cdot\infty\quad\text{(undefined)}$$

1.4 Absolute Numbers and Expressions

1.52. One `abs` function is equivalent to two functions, thus two equations

$$
|x-3|=7\quad\begin{cases}\underline{x-3\geq 0\Rightarrow x\geq 3} & \therefore\ |x-3|=x-3\ \ \therefore\ \ x-3=7\ \ \therefore\ \ \underline{x_1=10}\\ \underline{x-3<0\Rightarrow x<3} & \therefore\ |x-3|=-(x-3)\ \ \therefore\ \ -(x-3)=7\ \ \therefore\ \ \underline{x_2=-4}\end{cases}
$$

Therefore, there are two discrete solutions, $x_1=10$ if $(x\geq 3)$, and $x_2=-4$ if $(x<3)$, see Fig. 1.1.

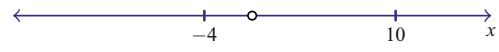


Fig. 1.1 Example P.1.52

1.53. An inequality with `abs` function is split into two inequalities, thus two *intervals* as the solutions,

$$|3-x|>4 \quad \begin{cases} \underline{3-x\geq 0 \Rightarrow x\leq 3} & \therefore\ |3-x|=3-x\ \ \therefore\ 3-x>4\ \ \therefore\ \underline{x<-1} \\ \underline{3-x<0 \Rightarrow x>3} & \therefore\ |3-x|=-(3-x)\ \ \therefore\ -(3-x)>4\ \ \therefore\ \underline{x>7} \end{cases}$$

Therefore, there are two intervals $x\in[-\infty,-1]$ and $x\in[7,+\infty]$, see Fig. 1.2.

1.54. There are two `abs` functions: $|x|$ and $||x|-5|$, thus in total there are four possible cases in

$$||x|-5|>4$$

systematically decomposed as follows:

$$\underline{x\geq 0}\ \Rightarrow |x|=x$$

$$\therefore\ |x-5|>4 \quad \begin{cases} \underline{x-5\geq 0}\ \Rightarrow |x-5|=x-5\ \ \therefore\ x-5>4 & \therefore\ \underline{x>9} \\ \underline{x-5<0}\ \Rightarrow |x-5|=-(x-5)\ \ \therefore\ -x+5>4 & \therefore\ \underline{x<1} \end{cases}$$

$$\underline{x<0}\ \Rightarrow |x|=-x$$

$$\therefore\ |-x-5|>4 \quad \begin{cases} \underline{-x-5\geq 0}\ \Rightarrow |-x-5|=-x-5\ \ \therefore\ -x-5>4 & \therefore\ \underline{x<-9} \\ \underline{-x-5<0}\ \Rightarrow |-x-5|=-(-x-5)\ \ \therefore\ x+5>4 & \therefore\ \underline{x>-1} \end{cases}$$

These four resulting inequalities must be satisfied at the same time, thus summarized as (see Fig. 1.3)

$$x\in[-\infty,-9] \quad \text{and} \quad -1\leq x\leq 1 \quad \text{and} \quad x\in[9,+\infty]$$

This conclusion my be verified by plotting $||x|-5|$ vs. "4." The three regions where $||x|-5|>4$ is satisfied are clearly visible, see Fig. 1.4.

1.55. Absolute function may be viewed as an efficient method to encode two equations

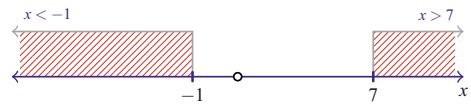


Fig. 1.2 Example P.1.53

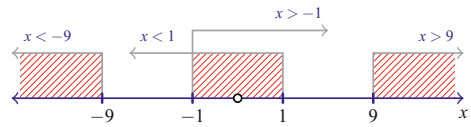
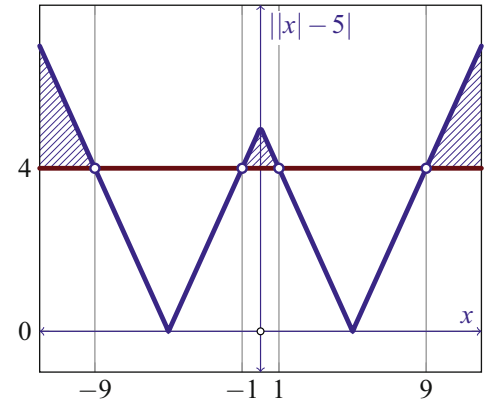


Fig. 1.3 Example P.1.54

Fig. 1.4 Example P.1.54



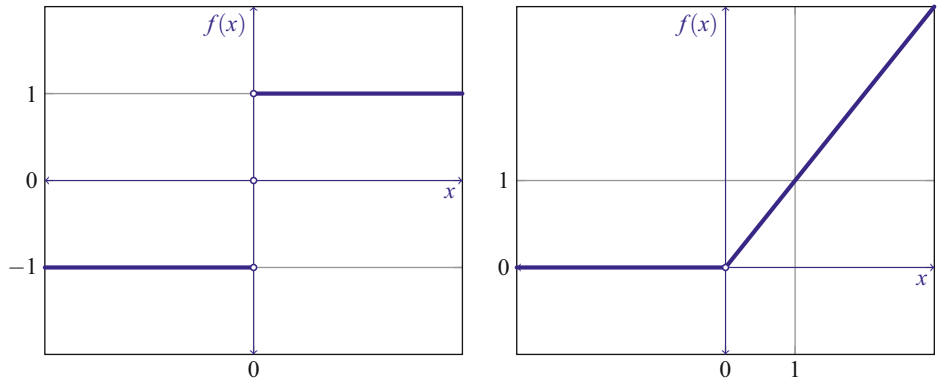


Fig. 1.5 Examples P.1.55, P.1.56

$$\frac{|x|}{x} = \left\{ \begin{array}{lll} \underline{x \geq 0} \;\Rightarrow |x| = x & \therefore & \dfrac{|x|}{x} = \dfrac{x}{x} = 1 \quad (x \neq 0) \\ \\ \underline{x < 0} \;:\Rightarrow |x| = -x & \therefore & \dfrac{|x|}{x} = \dfrac{-x}{x} = -1 \quad (x \neq 0) \end{array} \right\} = \operatorname{sign}(x)$$

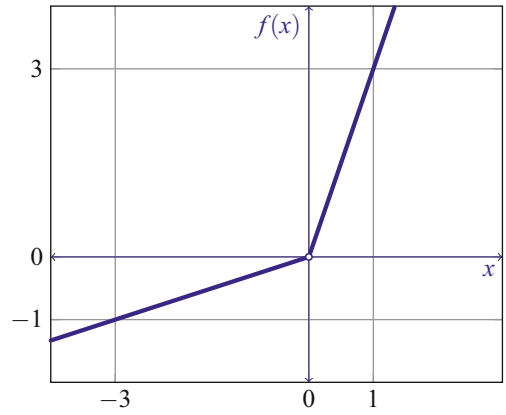
Indeed, this $|x|/x = x/|x| = \pm 1$ ratio is known as $\operatorname{sign}(x)$ function. By convention $\operatorname{sign}(0) = 0$, see Fig. 1.5 (left).

1.56. Signs of `abs` function's argument are

$$\frac{|x| + x}{2} = \left\{ \begin{array}{lll} \underline{x \geq 0} \;\Rightarrow |x| = x & \therefore & \dfrac{|x| + x}{2} = \dfrac{x + x}{2} = \dfrac{2x}{2} = x \\ \\ \underline{x < 0} \;\Rightarrow |x| = -x & \therefore & \dfrac{|x| + x}{2} = \dfrac{-x + x}{2} = \dfrac{0}{2} = 0 \end{array} \right\} = \operatorname{ramp}(x)$$

This is one of possible definitions for the function known as the $\operatorname{ramp}(x)$ function, see Fig. 1.5 (right).

Fig. 1.6 Example P.1.57



1.57. Absolute functions are replaced with their equivalent forms as

$$\frac{|x|+2x}{3}+|x|+x=\begin{cases}\underline{x\geq 0}\ \Rightarrow |x|=x & \therefore\ \dfrac{|x|+2x}{3}+|x|+x=\dfrac{x+2x}{3}+x+x=3x\\[2ex] \underline{x<0}\ \Rightarrow |x|=-x & \therefore\ \dfrac{|x|+2x}{3}+|x|+x=\dfrac{-x+2x}{3}-x+x=\dfrac{x}{3}\end{cases}$$

This result is an example of a general "piecewise linear" function, see Fig. 1.6. It is noted that there is a point where limits (and by extension, derivatives) are not defined (here, at $x=0$).

1.58. Given inequality

$$\frac{|x-|2x||}{|x|+x}-\frac{|x-2|}{|x|+1}>0$$

systematically decomposed as follows:

$$\underline{x\geq 0}\ \Rightarrow |x|=x\ \ \therefore\ \ |x|+x=x+x=2x\quad\text{and,}\quad |2x|=2x$$

$$\therefore\ \begin{cases}\underline{(x-2x)\geq 0}\ \Rightarrow |-x|\geq 0 & \therefore\ \underline{x\geq 0}\ \checkmark\\ \underline{(x-2x)<0}\ \Rightarrow |-x|<0 & \therefore\ \underline{\text{n.a.}}\end{cases}$$

$$\therefore$$

$$\frac{|x-|2x||}{|x|+x}=\frac{\cancel{x}\,1}{2\cancel{x}}=\frac{1}{2}$$

$$\underline{x<0}\ \Rightarrow |x|=-x\ \ \therefore\ \ |x|+x=-x+x=0\ \ \therefore\ \ \frac{|x-|2x||}{|x|+x}=\frac{|x-|2x||}{0}=\infty\ \ \therefore\ \ \underline{\text{n.a.}}$$

and,

$$\underline{x\geq 0}\ \Rightarrow |x|=x\ \ \therefore\ \ |x|+1=x+1$$

$$|x-2| = \begin{cases} \underline{(x-2) \geq 0} \Rightarrow |x-2| = x-2 & \therefore \ \underline{x \geq 2} \\ \underline{(x-2) < 0} \Rightarrow |x-2| = -(x-2) & \therefore \ \underline{x < 2} \end{cases}$$

In summary, as $x \geq 0$, (otherwise, inequality is not defined due to infinity term) then

(a) $\underline{0 \leq x \leq 2}$: note that within given interval $(x+1) > 0$

$$\begin{aligned} \frac{|x-|2x||}{|x|+x} - \frac{|x-2|}{|x|+1} &> 0 \\ \frac{1}{2} - \frac{-(x-2)}{x+1} &> 0 \\ \frac{(x+1)+2(x-2)}{2(x+1)} &> 0 \\ \frac{3(x-1)}{2(x+1)} &> 0 \quad \Rightarrow \underline{x > 1} \end{aligned}$$

(b) $\underline{x \geq 2}$: note that within given interval $(x+1) > 0$

$$\begin{aligned} \frac{|x-|2x||}{|x|+x} - \frac{|x-2|}{|x|+1} &> 0 \\ \frac{1}{2} - \frac{x-2}{x+1} &> 0 \\ \frac{(x+1)-2(x-2)}{2(x+1)} &> 0 \\ \frac{5-x}{2(x+1)} &> 0 \quad \Rightarrow \underline{x < 5} \end{aligned}$$

These sub-inequalities inequalities must be satisfied at the same time, see Fig. 1.7.

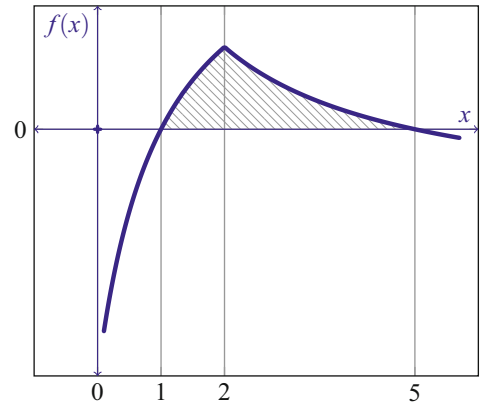
$$\frac{|x-|2x||}{|x|+x} - \frac{|x-2|}{|x|+1} > 0 \qquad \therefore \ \underline{x \in (1,5)}$$

1.59. Given expression

$$\left(\frac{x-2+|x-2|}{2}\right)^2 + \left(\frac{x-2-|x-2|}{2}\right)^2$$

Absolute function $|x-2|$ term is decomposed into two cases as

Fig. 1.7 Example P.1.58



1. $x-2\geq 0 \;\;\therefore\;\; x\geq 2 \;\Rightarrow |x-2|=(x-2)$

$$\left(\frac{x-2+|x-2|}{2}\right)^2+\left(\frac{x-2-|x-2|}{2}\right)^2=\left(\frac{x-2+(x-2)}{2}\right)^2+\left(\frac{x-2-(x-2)}{2}\right)^2$$
$$=\left[\frac{2(x-2)}{2}\right]^2+\left[\frac{x-2-x+2\;0}{2}\right]^2=(x-2)^2$$

2. $x-2<0 \;\;\therefore\;\; x<2 \;\Rightarrow |x-2|=-(x-2)$

$$\left(\frac{x-2+|x-2|}{2}\right)^2+\left(\frac{x-2-|x-2|}{2}\right)^2=\left(\frac{x-2-(x-2)}{2}\right)^2+\left(\frac{x-2+(x-2)}{2}\right)^2$$
$$=\left[\frac{x-2-x+2\;0}{2}\right]^2+\left[\frac{2(x-2)}{2}\right]^2=(x-2)^2$$

In summary,

$$\left(\frac{x-2+|x-2|}{2}\right)^2+\left(\frac{x-2-|x-2|}{2}\right)^2=(x-2)^2,\quad \forall x$$

Polynomials

2

Problems

2.1 Polynomial Expansion

Multiply polynomials in P.2.1 to P.2.6

2.1. $(x-1)(x^2+x+1)$

2.2. $(x-3)(x^2+3x+9)$

2.3. $(3x^2-5x+6)(2x-7)$

2.4. $(x^4+x^3+2)(x^2-3x+4)$

2.5. $(3y^2+2x-6xy)(2x^2+4xy-2y^3)$

2.6. $(x-y)(y-z)(z-x)$

Given $P(x)=x^2+x+1$ and $Q(x)=x^3-1$ in P.2.7 to P.2.9, calculate

2.7. $P(x)+Q(x)$

2.8. $P(x)-Q(x)$

2.9. $2xP(x)-2Q(x)$

Given $P(x)=x^3+ax^2+bx+c$ calculate parameters a, b, c so that $P(x)$ is divisible by binomials given in P.2.10 to P.2.13:

2.10. x, $(x-1)$, *and* $(x-2)$.

2.11. $(x+1)$, $(x-1)$, *and* $(x-2)$.

2.12. $(x+1)$, $(x-2)$, *and* $(x+3)$.

2.13. $(x-5)$, $(x+2)$, *and* $(x+4)$.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023
R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41200-4_2

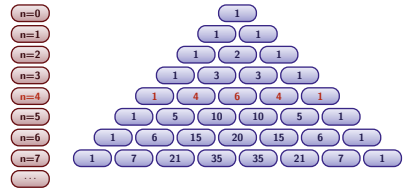


Fig. 2.1 Pascal's triangle

2.2 Binomial Theorem (Pascal's Triangle)

Reminder: Formal expression for binomial expansion is

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Due to their symmetry, the polynomial coefficients are arranged in the form of Pascal's triangle (Fig. 2.1).

Each row lists polynomial coefficients of the corresponding n-th binomial power. Note that each row coefficient is calculated as the sum of two neighboring coefficients, left and right in the row above.

For example, binomial square form is

$$(ax)^2 \pm 2abx + b^2 = (ax \pm b)^2$$

By using Pascal's triangle, develop binomial powers in P.2.14 to P.2.19.

2.14. $(a+b)^3$ **2.15.** $(xy-z)^5$ **2.16.** $(-a-bc)^4$

2.17. $(x+2)^5$ **2.18.** $(2x+1)^4$ **2.19.** $(x-1)^6$

Factorize quadratic and bi-quadratic polynomials given in P.2.20 to P.2.25.

2.20. $x^2 - 2x + 1$ **2.21.** $x^2 - 6x + 9$ **2.22.** $x^2 - 2\sqrt{2}x + 2$

2.23. $4x^2 - 12x + 9$ **2.24.** $x^4 - 2x^2 + 1$ **2.25.** $x^{2n+2} - 2x^{n+1} + 1$

2.3 Long Division

Divide polynomials in P.2.26 to P.2.29.

2.26. $(x^4 - 2x^3 - 7x^2 + 8x + 12) \div (x-3)$ **2.27.** $(6x^3 + 10x^2 + 8) \div (2x^2 + 1)$

2.28. $(4x^3 - 7x^2 - 11x + 5) \div (4x + 5)$ **2.29.** $(x^2 + x - 5) \div (x + 3)$

2.4 Factorization

Factorize polynomials in P.2.30 to P.2.41.

2.30. $5a + 5x$ **2.31.** $2a - 2$ **2.32.** $7a - 14$ **2.33.** $3a^2 + 9$

2.34. $3a + 6b + 9$ **2.35.** $6x + ax + bx$ **2.36.** $9a^2 - 6a + 12$ **2.37.** $a^2 - a^3$

2.38. $3a^2 - 6a$ **2.39.** $x^3a^2 - x^3$ **2.40.** $3a^3 + 2a^2 + a$ **2.41.** $4x^2 - 2x + xy$

Factorize polynomials in P.2.42 to P.2.53.

2.42. $x^3y^3 - x^3y + x^4y^3$ **2.43.** $x^3y^3 - x^2y^8$ **2.44.** $6x^2y^2 - 4xy^3$

2.45. $5x^3 - 15x^2y^3$ **2.46.** $6x^3y - 9x^2y^2 + 3x^3y^2$ **2.47.** $x^3 - x^7 - 2x^5$

2.48. $a^3b^2 + 2a^4b^2 - 4ab^5$ **2.49.** $3a^3b^3 - 9a^2b^4 + 12a^5b^4$ **2.50.** $a(m + n) + b(m + n)$

2.51. $m(a - b) + n(a - b)$ **2.52.** $7q(p - q) + 2p(q - p)$ **2.53.** $3(x + y) + (x + y)^2$

Factorize the following polynomials by finding the common terms: P.2.54 to P.2.63.

2.54. $am - an + bm - bn$ **2.55.** $am - an - bm + bn$ **2.56.** $an - ab - mn + mb$

2.57. $5ax + 5ay - x - y$ **2.58.** $2x^2 + 2xy - x - y$ **2.59.** $4ym - 4yn - m + n$

2.60. $ax^2 - bx^2 - bx + ax - a + b$ **2.61.** $6by - 15bx - 4ay + 10ax$

2.62. $2x^2 - 2xy + xz - yz$ **2.63.** $5ax^2 - 10ax - bx + 2b - x + 2$

Factorize polynomials in P.2.64 to P.2.65.

2.64. $xyz + x^2y^2 - 3x^4y^5 - 3x^3y^4 - xy - z$ **2.65.** $m^2x^4 - mnx^3 + 2mx^2 - 2nx + n - mx$

Factorize polynomials in P.2.66 to P.2.70.

2.66. $a^{2n} + a^n$ **2.67.** $a^{3x} - 2a^{2x}b^x$ **2.68.** $2x^{m+n} + 6x^n$

2.69. $a^{3x} + 3a^{2x} + 5a^{x}$ **2.70.** $x^{2n+2} - 2x^{n+1} + 1$

Simplify expressions in P.2.71 to P.2.73

2.71. $\dfrac{x^{n+2}}{2x\,x^{n}}$ **2.72.** $\dfrac{x^{3n+2} + x^{3n+1}}{x^{2}\,x^{3n}}$ **2.73.** $\dfrac{x + x^{2} + x^{3} + \cdots + x^{n}}{\frac{1}{x} + \frac{1}{x^{2}} + \frac{1}{x^{3}} + \cdots + \frac{1}{x^{n}}}$

2.5 Difference of Squares

Reminder: Difference of two squares identity is used in many practical sitautions

$$a^{2} - b^{2} = (a - b)(a + b)$$

Factorize polynomials in P.2.74 to P.2.89.

2.74. $x^{2} - 49$ **2.75.** $a^{2} - 36$ **2.76.** $16x^{2} - 9$ **2.77.** $9x^{2} - 49$

2.78. $25 - x^{2}$ **2.79.** $81 - x^{4}$ **2.80.** $x^{2} - \dfrac{1}{49}$ **2.81.** $\dfrac{x^{2}}{4} - \dfrac{4}{9}$

2.82. $\dfrac{9x^{2}}{4} - \dfrac{4y^{2}}{9}$ **2.83.** $\dfrac{49x^{2}}{25} - 9y^{2}$ **2.84.** $x^{2} - 0.36$ **2.85.** $x^{2} - 0.0009$

2.86. $0.04x^{2} - 0.25$ **2.87.** $0.01x^{2} - 0.04y^{2}$ **2.88.** $x^{4}y^{2} - 0.01$ **2.89.** $0.25x^{2}y^{2} - 0.0001$

Factorize expressions in P.2.90 to P.2.101.

2.90. $(x - 3)^{2} - 4$ **2.91.** $(a + 5)^{2} - 9$ **2.92.** $y^{2} - (x - y)^{2}$

2.93. $x^{2} - (x + y)^{2}$ **2.94.** $(x + 2)^{2} - 4x^{2}$ **2.95.** $9x^{2} - (x - 1)^{2}$

2.96. $(x - y)^{2} - 16(x + y)^{2}$ **2.97.** $(x + 2y)^{2} - 9(x - 2y)^{2}$ **2.98.** $4(x - y)^{2} - 25(x + y)^{2}$

2.99. $36(x - 2)^{2} - 25(x + 1)^{2}$ **2.100.** $(x + y - z)^{2} - (x - y + z)^{2}$ **2.101.** $(x + y - 3)^{2} - 25(x + 2)^{2}$

Calculate products in P.2.102 to P.2.110 *without using a calculator*.

2.102. 98×102 **2.103.** 99×101 **2.104.** 83×77 **2.105.** 79×81

2.106. 18×22 **2.107.** 201×199 **2.108.** 1.05×0.95

2.109. 1.01×0.99 **2.110.** 9.9×10.1

2.6 Quadratic Polynomial—Viète Formulas

Reminder: The relations between the sum and product of polynomial roots, attributed to François Viète, are rather practical, among other applications in mathematics, for rapid factorization of quadratic polynomials. Trinomial

$$ax^2 + bx + c = x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 + b'x + c'$$

may be factorized as

$$x^2 + b'x + c' = x^2 + x_1 x + x_2 x + x_1 x_2 = x(x + x_1) + x_2(x + x_1) = (x + x_1)(x + x_2)$$

where, x_1, x_2 are factors of c' and $x_1 + x_2 = b'$

Factorize quadratic polynomials in P.2.111 to P.2.119.

2.111. $x^2 - 6x + 5$ **2.112.** $x^2 - 9x + 14$ **2.113.** $x^2 - 6x + 8$

2.114. $2x^2 + 3x + 1$ **2.115.** $3x^2 - 14x + 8$ **2.116.** $-2x^2 + x + 3$

2.117. $-x^2 + 5x - 4$ **2.118.** $-6x^2 + 5x + 4$ **2.119.** $x^2 + (\sqrt{2} + \sqrt{3})x + \sqrt{6}$

Factorize bi-quadratic polynomials in P.2.120 to P.2.125.

2.120. $x^4 - 13x^2 + 36$ **2.121.** $x^4 - 10x^2 + 9$ **2.122.** $x^6 - 2x^3 + 1$

2.123. $x^6 + 3x^3 + 2$ **2.124.** $x^6 - 9x^3 + 8$ **2.125.** $2x^{2m+2} - 11x^{m+1} + 9$

2.7 Completing the Square

Reminder: Technique of completing the square helps convert a quadratic polynomial of the form

$$ax^2 + bx + c \quad \text{to the form} \quad a(x - m)^2 + n$$

where m and n are some suitable constants.

That is to say, completing the square creates a square trinomial inside of a quadratic expression.

Solve quadratic equations in P.2.126 to P.2.131.

2.126. $x^2 + 6x - 7 = 0$

2.127. $2x^2 - 10x - 3 = 0$

2.128. $-x^2 - 6x + 7 = 0$

2.129. $x^2 + 3x = 0$

2.130. $2x^2 + 6x + 2 = 0$

2.131. $x^2 - x - 6 = 0$

2.8 Factor Theorem

Reminder: Given n-th-order polynomial

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = (x - x_n)(x - x_{n-1}) \cdots (x - x_1)$$

where, $x_n, x_{n-1}, \ldots, x_1$, are zeros of $P_n(x)$ (in total, there are n zeros).

1. Factorization of polynomials takes advantage of fundamental theorems in algebra (very loosely interpreted as):
   (a) The total number of polynomial roots n (i.e., both real and complex) is equal to the polynomial degree n, where the complex roots come in pairs, i.e., each complex root is accompanied by its complex conjugate pair. Consequently, in odd order polynomials, there must be at least one real root (i.e., the one that does not have its pair).
   (b) Polynomial roots may be found among the factors of the $(a_n\, a_0)$ product.
   (c) If $(x - x_0)$ is one of $P_n(x)$ factors, that is, x_0 is one of $P_n(x)$ zeros, then $P_n(x_0) = 0$.
2. The factor theorem states that $(x - n)$ is binomial factor of a polynomial $P(x)$ if and only if $P(n) = 0$, that is, n is one of the $P(x)$ roots. Otherwise, if the value $P(n) = r \neq 0$, it is said that $R = r/(x - n)$ is the reminder of division $P(x) \div (x - n)$.

Factorize: P.2.132 to P.2.143

2.132. $x^3 - 1$ **2.133.** $x^3 - y^3$ **2.134.** $x^3 + 1$ **2.135.** $x^3 + x + 2$

2.136. $x^3 + y^3$ **2.137.** $x^4 + 4$ **2.138.** $x^4 + x^2 + 1$ **2.139.** $x^5 + x + 1$

2.140. $(x + y + z)^3 - x^3 - y^3 - z^3$ **2.141.** $x^3 + x^2 - 4x - 4$

2.142. $(x^2 + x + 1)(x^3 + x^2 + 1) - 1$ **2.143.** $(x + 1)(x + 3)(x + 5)(x + 7) + 15$

Calculate parameters a, b, c so that $P(x) = Q(x)$ in P.2.144 to P.2.145, if

2.144. $P(x) = x^3 - 2x^2 + 3, \quad Q(x) = (x + 1)(ax^2 + bx + c)$

2.145. $P(x) = 2x^3 - x^2 + x + 4, \quad Q(x) = (x - 2)(ax^2 + bx + c)$

Without doing the actual division, calculate the reminder of polynomial divisions in P.2.146 to P.2.149.

2.146. $(x^4 - 2x^3 + 3x^2 - 4x + 1) \div (x - 1)$ **2.147.** $(x^3 + 1) \div (x + 1)$

2.148. $\left(x^{1965} - 256x^{1961} + 1\right) \div \left(x^2 - 4x\right)$ **2.149.** $\left(x^{2023} - 2x^{2022} - 1\right) \div \left(x^2 - 3x + 2\right)$

Given polynomials in P.2.150 to P.2.151, calculate parameter k so that $P(x, k)$ is divisible by $Q(x)$.

2.150. $P(x, k) = 4x^5 + k\,x^4 + 8x^3 + 5x^2 + 3x + 2, \quad Q(x) = x + 2.$

2.151. $P(x, k) = x^3 - 3k\,x + 4(k^2 + 1)x - (k^3 + 5), \quad Q(x) = (x - 1)$

2.9 Partial Fraction Decomposition

Reminder: Partial fraction form of a rational function is very useful for calculating its integral. A general case of rational expression of two polynomials, for example, k-th order $P_k(x)$ and m-th order $Q_m(x)$, where $m > k$, may be written as

$$\frac{P_k(x)}{Q_m(x)} = \frac{P(x)}{(x - x_1)(x - x_2)(x - x_3)^n(ax^2 + bx + c)}$$
$$= \frac{\alpha}{x - x_1} + \frac{\beta}{x - x_2} + \frac{\gamma_1}{(x - x_3)^1} + \frac{\gamma_2}{(x - x_3)^2} + \cdots + \frac{\gamma_n}{(x - x_3)^n} + \frac{\delta x + \eta}{(ax^2 + bx + c)}$$

where, $\alpha, \beta, \gamma, \ldots$ are the coefficient constants to be calculated, and partial decomposition rational forms are created as follows:

1. For each unique zero of $Q(x)$, such as x_1, x_2, there is one rational term.

(continued)

2. For zeros with n multiplicity, such as x_3, there is a series of n rational terms.
3. When quadratic term $(ax^2 + bx + c)$ has complex zeros, then its corresponding numerator must be linear binomial.

Derive the partial fraction form of rational functions in P.2.152 to P.2.166.

2.152. $\dfrac{x}{(x+1)(x-4)}$

2.153. $\dfrac{x+2}{x^3-2x^2}$

2.154. $\dfrac{2x^2}{x^4-1}$

2.155. $\dfrac{1}{x^3-1}$

2.156. $\dfrac{x^3+x-1}{(x^2+2)^2}$

2.157. $\dfrac{4}{x^4+1}$

2.158. $\dfrac{x^3+x^2-16x+16}{x^2-4x+3}$

2.159. $\dfrac{x-1}{x^2+x}$

2.160. $\dfrac{x}{(x+1)(x-4)}$

2.161. $\dfrac{1}{x^2-1}$

2.162. $\dfrac{x+2}{x^3-2x^2}$

2.163. $\dfrac{2x^2}{x^4-1}$

2.164. $\dfrac{1}{x^3-1}$

2.165. $\dfrac{4}{x^4+1}$

2.166. $\dfrac{1}{(x+1)(x+2)}$

Answers

2.1 Polynomials

2.1. $(x-1)(x^2+x+1) = x^3 + \cancel{x^2} + \cancel{x} - \cancel{x^2} - \cancel{x} - 1 = x^3 - 1$

2.2. $(x-3)(x^2+3x+9) = x^3 + \cancel{3x^2} + \cancel{9x} - \cancel{3x^2} - \cancel{9x} - 27 = x^3 - 27 = x^3 - 3^3$

2.3. $(3x^2-5x+6)(2x-7) = 6x^3 - 21x^2 - 10x^2 + 35x + 12x - 42 = 6x^3 - 31x^2 + 47x - 42$

2.4. $(x^4+x^3+2)(x^2-3x+4) = x^6 - 3x^5 + 4x^4 + x^5 - 3x^4 + 4x^3 + 2x^2 - 6x + 8$
$= x^6 - 2x^5 + x^4 + 4x^3 + 2x^2 - 6x + 8$

2.5. $(3y^2 + 2x - 6xy)(2x^2 + 4xy - 2y^3)$
$= 6x^2y^2 + 12xy^3 - 6y^5 + 4x^3 + 8x^2y - 4xy^3 - 12x^3y - 24x^2y^2 + 12xy^2$
$= -18x^2y^2 + 8xy^3 - 6y^5 + 4x^3 + 8x^2y - 12x^3y + 12xy^4$

2.6. $(x-y)(y-z)(z-x) = (xy - xz - y^2 + yz)(z-x)$
$= \cancel{xyz} - xz^2 - y^2z + yz^2 - x^2y + x^2z + xy^2 - \cancel{xyz}$
$= x^2z + yz^2 + xy^2 - xz^2 - y^2z - x^2y$

2.7. Given $P(x) = x^2 + x + 1$ and $Q(x) = x^3 - 1$

$$P(x) + Q(x) = (x^2+x+1) + (x^3-1) = x^3 - \cancel{1} + x^2 + x + \cancel{1} = x(x^2+x+1)$$

2.8. Given $P(x) = x^2 + x + 1$ and $Q(x) = x^3 - 1$

$$P(x) - Q(x) = (x^2+x+1) - (x^3-1) = -x^3 + 1 + x^2 + x + 1 = -x^3 + x^2 + x + 2$$

2.9. Given $P(x) = x^2 + x + 1$ and $Q(x) = x^3 - 1$

$$2xP(x) - 2Q(x) = 2x(x^2+x+1) - 2(x^3-1) = \cancel{2x^3} + 2x^2 + 2x - \cancel{2x^3} + 2$$
$$= 2x^2 + 2x + 2 = 2(x^2+x+1)$$

2.10. Third-order polynomial $Q(x)$ that is divisible by x, $(x-1)$, and $(x-2)$ must have form

$$Q(x) = (x+0)(x-1)(x-2) = x(x^2-3x+2) = x^3 - 3x^2 + 2x$$

Therefore,

$$P(x) = Q(x) \quad \therefore \quad x^3 + ax^2 + bx + c = x^3 - 3x^2 + 2x \quad \therefore \quad a = -3,\ b = 2, \quad \text{and} \quad c = 0$$

2.11. Third-order polynomial $Q(x)$ that is divisible by $(x+1)$, $(x-1)$, and $(x+2)$ must have form

$$Q(x) = (x+1)(x-1)(x+2) = (x^2-1)(x+2) = x^3 + 2x^2 - x - 2$$

Therefore,

$$P(x) = Q(x) \;\;\therefore\;\; x^3 + ax^2 + bx + c = x^3 + 2x^2 - x - 2 \;\;\therefore\;\; a = 2,\; b = -1, \quad \text{and} \quad c = -2$$

2.12. Third-order polynomial $Q(x)$ that is divisible by $(x+1)$, $(x-2)$, and $(x+3)$ must have form

$$Q(x) = (x+1)(x-2)(x+3) = x^3 + 2x^2 - 5x - 6$$

Therefore,

$$P(x) = Q(x) \;\;\therefore\;\; x^3 + ax^2 + bx + c = x^3 + 2x^2 - 5x - 6 \;\;\therefore\;\; a = 2,\; b = -5, \quad \text{and} \quad c = -6$$

2.13. Third-order polynomial $Q(x)$ that is divisible by $(x-5)$, $(x+2)$, and $(x+4)$ must have form

$$Q(x) = (x-5)(x+2)(x+4) = x^3 + x^2 - 22x - 40$$

Therefore,

$$P(x) = Q(x) \;\;\therefore\;\; x^3 + ax^2 + bx + c = x^3 + x^2 - 22x - 40 \;\;\therefore\;\; a = 1,\; b = -22, \quad \text{and} \quad c = -40$$

2.2 Binomial Theorem (Pascal's Triangle)

2.14. Given $(a+b)^3$, coefficients of the third-order binomial power are listed in Pascal's triangle as 1, 3, 3, 1. Then, the powers of a and b are systematically written in the descending and ascending orders, respectively, as

$$(a+b)^3 = 1\,a^3\,b^0 + 3\,a^2\,b^1 + 3\,a^1\,b^2 + 1\,a^0\,b^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

2.15. Given $(xy - z)^5$, coefficients of the fifth-order binomial power are listed in Pascal's triangle as 1, 5, 10, 10, 5, 1. Then,

$$\begin{aligned}(xy - z)^5 &= 1\,(xy)^5\,(-z)^0 + 5\,(xy)^4\,(-z)^1 + 10\,(xy)^3\,(-z)^2 + 10\,(xy)^2\,(-z)^3 \\ &\quad + 5\,(xy)^1\,(-z)^4 + 1\,(xy)^0\,(-z)^5 \\ &= x^5y^5 - 5zx^4y^4 + 10z^2x^3y^3 - 10z^3x^2y^2 + 5z^4xy - z^5\end{aligned}$$

2.16. Given $(-a-bc)^4$, coefficients of the fourth-order binomial power are listed in Pascal's triangle as 1, 4, 6, 4, 1. Then,

$$\begin{aligned}&(-a - bc)^4 \\ &\quad = 1\,(-a)^4\,(-bc)^0 + 4\,(-a)^3\,(-bc)^1 + 6\,(-a)^2\,(-bc)^2 + 4\,(-a)^1\,(-bc)^3 + 1\,(-a)^0\,(-bc)^4 \\ &\quad = a^4 + 4a^3bc + 6a^2b^2c^2 + 4ab^3c^3 + b^4c^4\end{aligned}$$

2.17. $(x+2)^5 = 1\,x^5\,2^0 + 5\,x^4\,2^1 + 10\,x^3\,2^2 + 10\,x^2\,2^3 + 5\,x^1\,2^4 + 1\,x^0\,2^5$
$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

2.18. $(2x+1)^4 = 1\,(2x)^4\,1^0 + 4\,(2x)^3\,1^1 + 6\,(2x)^2\,1^2 + 4\,(2x)^1\,1^3 + 1\,(2x)^0\,1^4$
$= 16x^4 + 32x^3 + 24x^2 + 8x + 1$

2.19. $(x-1)^6 = 1\,x^6\,(-1)^0 + 6\,x^5\,(-1)^1 + 15\,x^4\,(-1)^2 + 20\,x^3\,(-1)^3 +$
$15\,x^2\,(-1)^4 + 6\,x^1\,(-1)^5 + 1\,x^0\,(-1)^6$
$= x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$

2.20. $x^2 - 2x + 1 = (1x)^2 - 2\,(1)\,(1)x + 1^2 = \{n=2\} = (x-1)^2$

2.21. $x^2\underline{-6x} + 9 = x^2\underline{-2\cdot 3x} + 3^2 = (x-3)^2$

2.22. $x^2 - 2\sqrt{2}x + \underline{2} = x^2 - 2\sqrt{2}x + \underline{(\sqrt{2})^2} = (x-\sqrt{2})^2$

2.23. $4x^2 - 12x + 9 = (2x)^2 - 2\cdot(3\cdot 2x) + 3^2 = (2x-3)^2$

2.24. $x^4 - 2x^2 + 1 = (x^2)^2 - 2x^2 + 1 = (x^2-1)^2 = (x+1)^2(x-1)^2$

2.25. $x^{2n+2} - 2x^{n+1} + 1 = x^{2(n+1)} - 2x^{n+1} + 1 = \left(x^{n+1}\right)^2 - 2\left(x^{n+1}\right) + 1 = (x^{n+1}-1)^2$

2.3 Long Division

2.26. Division of two polynomials follows the same procedure as division of any two large numbers. There are multiple techniques to organize the polynomial coefficients and visual tabulation of the intermediate results. Here, one possible way is to first rearrange all terms of the two given polynomials in descending powers. Then, if the last step results in zero it is said that two polynomials are divisible, otherwise, there is reminder r. Step by step procedure may be as,

1. Divide only the two highest power terms terms (underlined) and write the result, that is to say $x^4 \div x = x^3$, so that

$$(\underline{x^4} - 2x^3 - 7x^2 + 8x + 12) \div (\underline{x} - 3) = \underline{x^3}$$

2. Multiply x^3 with the divisor polynomial $(x-3)$ and write the product under dividend

$$(x^4 - 2x^3 - 7x^2 + 8x + 12) \div \underline{(x-3)} = \underline{x^3}$$
$$\underline{x^4 - 3x^3} \qquad \text{Multiply}$$

3. Subtract terms that are aligned vertically, i.e., $(x^4 - 2x^3) - (x^4 - 3x^2) = x^3$, then add the next term from dividend, i.e., "$-7x^2$." Now, division is $(x^3 - 7x^2) \div (x - 3)$ as

$$\begin{aligned} &(x^4 - 2x^3 - 7x^2 + 8x + 12) \div (x - 3) = x^3 \\ &-(x^4 - 3x^3) \\ &= \ 0 \ + x^3 \ -7x^2 \end{aligned}$$

4. Next term in the solution polynomial is found as $x^3 \div x = x^2$, then repeat the previous steps until the last term in the solution is found as

$$\begin{aligned} &(x^4 - 2x^3 - 7x^2 + 8x + 12) \div (x - 3) = x^3 + x^2 - 4x - 4 \\ &-(x^4 - 3x^3) \\ &\qquad x^3 - 7x^2 \\ &\quad -(x^3 - 3x^2) \\ &\qquad -4x^2 + 8x \\ &\quad -(-4x^2 + 12x) \\ &\qquad\qquad -4x + 12 \\ &\qquad\quad -(-4x + 12) \\ &\qquad\qquad\ \ = 0 \end{aligned}$$

Because the last subtraction equals zero means that $r = 0$, that is to say $x = 3$ is zero of $P(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$. It is verified by setting $x = 3$ in $P(x)$ as

$$P(3) = (3)^4 - 2(3)^3 - 7(3)^2 + 8(3) + 12 = 81 - 54 - 63 + 24 + 12 = 0$$

2.27. Given $(6x^3 + 10x^2 + 8) \div (2x^2 + 1)$, long division gives,

$$(6x^3 + 10x^2 + 0x + 8) \div (2x^2 + 1) = 3x + 5 + \underbrace{\frac{-3x + 3}{2x^2 + 1}}_{r}$$

$$\begin{aligned} &6x^3 \ +0x^2 + 3x \\ &\qquad 10x^2 - 3x + 8 \\ &\qquad 10x^2 \ +0x + 5 \\ &\qquad\ \ -3x + 3 \neq 0 \end{aligned}$$

2.28. Given $(4x^3 - 7x^2 - 11x + 5) \div (4x + 5)$, long division gives,

$$(4x^3 - 7x^2 - 11x + 5) \div (4x + 5) = x^2 - 3x + 1$$
$$\underline{4x^3 + 5x^2}$$
$$\quad -12x^2 - 11x$$
$$\quad \underline{-12x^2 - 15x}$$
$$\qquad\qquad 4x + 5$$
$$\qquad\qquad \underline{4x + 5}$$
$$\qquad\qquad\quad = 0$$

2.29. Given $(x^2 + x - 5) \div (x + 3)$, long division gives,

$$(x^2 \quad + x - 5) \div (x + 3) = x - 2 + \underbrace{\frac{1}{x+3}}_{r}$$
$$\underline{x^2 + 3x}$$
$$\quad -2x - 5$$
$$\quad \underline{-2x - 6}$$
$$\qquad = 1$$

2.4 Factorization

2.30. $\underline{5}\, a + \underline{5}\, x = \underline{5}\, (a + x)$

2.31. $2a - 2 = 2(a - 1)$

2.32. $7a - 14 = 7(a - 2)$

2.33. $3a^2 + 9 = 3(a^2 + 3)$

2.34. $3a + 6b + 9 = 3(a + 2b + 3)$

2.35. $6x + ax + bx = x(6 + a + b)$

2.36. $9a^2 - 6a + 12 = 3(3a^2 - 2a + 4)$

2.37. $a^2 - a^3 = a^2(1 - a)$

2.38. $3a^2 - 6a = 3a(a - 2)$

2.39. $x^3a^2 - x^3 = x^3(a^2 - 1) = x^3(a + 1)(a - 1)$

2.40. $3a^3 + 2a^2 + a = a(3a^2 + 2a + 1)$

2.41. $4x^2 - 2x + xy = x(4x - 2 - y)$

2.42. $x^3y^3 - x^3y + x^4y^3 = \underline{x^3}\ \underline{y}\ y^2 - \underline{x^3}\ \underline{y}\ + \underline{x^3}\ x\ \underline{y}\ y^2 = \underline{x^3}\ \underline{y}\ (y^2 - 1 + xy^2)$

2.43. $x^3y^3 - x^2y^8 = x^2y^3(x - y^5)$

2.44. $6x^2y^2 - 4xy^3 = 2xy^2(3x - 2y)$

2.45. $5x^3 - 15x^2y^3 = 5x^2(x - 3y^3)$

2.46. $6x^3y - 9x^2y^2 + 3x^3y^2 = 3x^2y(2x - 3y + xy)$

2.47. $x^3 - x^7 - 2x^5 = x^3(1 - x^4 - 2x^2)$

2.48. $a^3b^2 + 2a^4b^2 - 4ab^5 = ab^2(a^2 + 2a^3 - 4b^3)$

2.49. $3a^3b^3 - 9a^2b^4 + 12a^5b^4 = 3a^2b^3(a - 3b + 4a^3b)$

2.50. $a(m + n) + b(m + n) = (m + n)(a + b)$

2.51. $m(a - b) + n(a - b) = (a - b)(m + n)$

2.52. $7q(p - q) + 2p(q - p) = (p - q)(7q - 2p)$

2.53. $3(x + y) + (x + y)^2 = (x + y)(3 + x + y)$

2.54. $\underline{a}\ m - \underline{a}\ n + \underline{b}\ m - \underline{b}\ n = \underline{a}\ \underline{(m - n)}\ + \underline{b}\ \underline{(m - n)}\ = \underline{(m - n)}\ (a + b)$

2.55. $am - an - bm + bn = a(m - n) - b(m - n) = (m - n)(a - b)$

2.56. $an - ab - mn + mb = a(n - b) - m(n - b) = (n - b)(a - m)$

2.57. $5ax + 5ay - x - y = 5a(x + y) - (x + y) = (x + y)(5a - 1)$

2.58. $2x^2 + 2xy - x - y = 2x(x + y) - (x + y) = (x + y)(2x - 1)$

2.59. $4ym - 4yn - m + n = 4y(m - n) - (m - n) = (m - n)(4y - 1)$

2.60. $ax^2 - bx^2 - bx + ax - a + b = x^2(a - b) + x(a - b) - (a - b) = (a - b)(x^2 + x - 1)$

2.61. $6by - 15bx - 4ay + 10ax = 2y(3b - 2a) - 5x(3b - 2a) = (3b - 2a)(2y - 5x)$

2.62. $2x^2 - 2xy + xz - yz = 2x(x - y) + z(x - y) = (x - y)(2x + z)$

2.63. $5ax^2 - 10ax - bx + 2b - x + 2 = 5ax(x-2) - b(x-2) - (x-2)$
$= (x-2)(5ax - b - 1)$

2.64. $xyz + x^2y^2 - 3x^4y^5 - 3x^3y^4z - xy - z = xy(xy+z) - 3x^3y^4(xy+z) - (xy+z)$
$= (xy+z)(xy - 3x^3y^4 - 1)$

2.65. $m^2x^4 - mnx^3 + 2mx^2 - 2nx + n - mx = mx^3(mx-n) + 2x(mx-n) - (mx-n)$
$= (mx-n)(mx^3 + 2x - 1)$

2.66. $a^{2n} + a^n = \underline{a^n}\ a^n + \underline{a^n} = \underline{a^n}\,(a^n+1)$

2.67. $a^{3x} - 2a^{2x}b^x = a^{2x}(a^x - 2b^x)$

2.68. $2x^{m+n} + 6x^n = 2x^n(x^m + 3)$

2.69. $a^{3x} + 3a^{2x} + 5a^x = a^x(a^{2x} + 3a^x + 5)$

2.70. $x^{2n+2} - 2x^{n+1} + 1 = x^{2(n+1)} - 2x^{n+1} + 1 = \left(x^{n+1}\right)^2 - 2\left(x^{n+1}\right) + 1 = \left(x^{n+1} - 1\right)^2$

2.71. $\dfrac{x^{n+2}}{2x\,x^n} = \dfrac{x\,\cancel{x^{n+1}}}{2\cancel{x^{n+1}}} = \dfrac{x}{2}$

2.72. $\dfrac{x^{3n+2} + x^{3n+1}}{x^2\,x^{3n}} = \dfrac{\cancel{x^{3n+1}}\,(x+1)}{x\,\cancel{x^{3n+1}}} = \dfrac{x+1}{x}$

2.73. $\dfrac{x + x^2 + x^3 + \cdots + x^n}{\dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \cdots + \dfrac{1}{x^n}} = \dfrac{x + x^2 + x^3 + \cdots + x^n}{\dfrac{x^{n-1} + x^{n-2} + x^{n-3} + \cdots + 1}{x^n}}$
$= \dfrac{\left(x + x^2 + x^3 + \cdots + x^n\right)x^n}{x^{n-1} + x^{n-2} + x^{n-3} + \cdots + 1}$
$= \dfrac{\left(1 + x^2 + x^2 + \cdots + x^{n-1}\right)x^{n+1}}{\cancel{x^{n-1} + x^{n-2} + x^{n-3} + \cdots + 1}} = x^{n+1}$

2.5 Difference of Squares

2.74. $x^2 - 49 = x^2 - 7^2 = (x+7)(x-7)$

2.75. $a^2 - 36 = a^2 - 6^2 = (a+6)(a-6)$

2.76. $16x^2 - 9 = (4x)^2 - 3^2 = (4x+3)(4x-3)$

2.77. $9x^2 - 49 = (3x)^2 - 7^2 = (3x+7)(3x-7)$

2.78. $25 - x^2 = 5^2 - x^2 = (5+x)(5-x)$

2.79. $81 - x^4 = 9^2 - (x^2)^2 = (9 + x^2)(9 - x^2) = (9 + x^2)(3 + x)(3 - x)$

2.80. $x^2 - \frac{1}{49} = \left(x + \frac{1}{7}\right)\left(x - \frac{1}{7}\right)$

2.81. $\frac{x^2}{4} - \frac{4}{9} = \left(\frac{x}{2}\right)^2 - \left(\frac{2}{3}\right)^2 = \left(\frac{x}{2} + \frac{2}{3}\right)\left(\frac{x}{2} - \frac{2}{3}\right)$

2.82. $\frac{9x^2}{4} - \frac{4y^2}{9} = \left(\frac{3x}{2} + \frac{2y}{3}\right)\left(\frac{3x}{2} - \frac{2y}{3}\right)$

2.83. $\frac{49x^2}{25} - 9y^2 = \left(\frac{7x}{5}\right)^2 - (3y)^2 = \left(\frac{7x}{5} + 3\right)\left(\frac{7x}{5} - 3\right)$

2.84. $x^2 - 0.36 = x^2 - \frac{\cancel{9}\ 36}{\cancel{25}\ 100} = x^2 - \left(\frac{3}{5}\right)^2 = \left(x + \frac{3}{5}\right)\left(x - \frac{3}{5}\right)$

2.85. $x^2 - 0.0009 = x^2 - 0.03^2 = (x + 0.03)\,(x - 0.03)$

2.86. $0.04x^2 - 0.25 = (0.2x)^2 - 0.5^2 = (0.2x + 0.5)(0.2x - 0.5) = \frac{1}{100}(2x + 5)(2x - 5)$

2.87. $0.01x^2 - 0.04y^2 = (0.1x)^2 - (0.2y)^2 = (0.1x + 0.2y)(0.1x - 0.2y)$
$= \frac{1}{100}(x + 2y)(x - 2y)$

2.88. $x^4y^2 - 0.01 = \left(x^2y\right)^2 - 0.1^2 = (x^2y + 0.1)(x^2y - 0.1)$

2.89. $0.25x^2y^2 - 0.0001 = (0.5xy + 0.01)(0.5xy - 0.01)$

2.90. $(x - 3)^2 - 4 = (x - 3 + 2)(x - 3 - 2) = (x - 1)(x - 5)$

2.91. $(a + 5)^2 - 9 = (a + 5 + 3)(a + 5 - 3) = (a + 8)(a + 2)$

2.92. $y^2 - (x - y)^2 = (\cancel{y} + x - \cancel{y})(y - (x - y)) = x(2y - x)$

2.93. $x^2 - (x + y)^2 = (x + x + y)(\cancel{x} - \cancel{x} - y) = -y(2x + y)$

2.94. $(x + 2)^2 - 4x^2 = (x + 2 + 2x)(x + 2 - 2x) = (3x + 2)(2 - x)$

2.95. $9x^2 - (x - 1)^2 = (3x + x - 1)(3x - (x - 1)) = (4x - 1)(2x + 1)$

2.96. $(x - y)^2 - 16(x + y)^2 = (x - y + 4x + 4y)(x - y - 4x - 4y)$
$= (5x + 3y)(-3x - 5y) = -(5x + 3y)(3x + 5y)$

2.97. $(x+2y)^2-9(x-2y)^2=(x+2y+3x-6y)(x+2y-(3x-6y))$
$=(4x-4y)(8y-2x)=4(x-y)\,2(4y-x)$
$=8\,(x-y)(4y-x)$

2.98. $4(x-y)^2-25(x+y)^2=(2x-2y+5x+5y)(2x-2y-5x-5y)$
$=(7x+3y)(-3x-7y)=-(7x+3y)(3x+7y)$

2.99. $36(x-2)^2-25(x+1)^2=(6x-12+5x+5)(6x-12-5x-5)=(11x-7)(x-17)$

2.100. $(x+y-z)^2-(x-y+z)^2=(x+y-z+x-y+z)(x+y-z-x+y-z)$
$=2x(2y-2z)=4x(y-z)$

2.101. $(x+y-3)^2-25(x+2)^2=(x+y-3+5x+10)(x+y-3-5x-10)$
$=(6x+y+7)(y-4x-13)$

2.102. $98\times102=(100-2)(100+2)=100^2-2^2=10{,}000-4=9996$

2.103. $99\times101=(100-1)(100+1)=100^2-1=9\,999$

2.104. $83\times77=(80+3)(80-3)=80^2-3^2=6\,400-9=6\,391$

2.105. $79\times81=(80-1)(80+1)=80^2-1=6\,400-1=6\,399$

2.106. $18\times22=(20-2)(20+2)=400-4=396$

2.107. $201\times199=(200+1)(200-1)=40\,000-1=39\,999$

2.108. $1.05\times0.95=(1+0.05)(1-0.05)=1-0.05^2=1-0.0025=0.9975$

2.109. $1.01\times0.99=(1+0.01)(1-0.01)=1-0.01^2=1-0.0001=0.9999$

2.110. $9.9\times10.1=(10-0.1)(10+0.1)=100-0.01=99.99$

2.6 Quadratic Polynomial—Viète Formulas

2.111. Given x^2-6x+5, in accordance to Viète formulas the goal is to check existence of x_1, x_2 pair so that

$$x_1\,x_2=5 \quad \text{and} \quad x_1+x_2=6$$

Being prime number, 'five' is divisible only with $\pm1, \pm5$. Therefore, its factors -1 and -5 satisfy both equalities, i.e. their product equals $+5$ and their sum is -6. Thus,

$$x^2\underline{-6x}\ +5=x^2\underline{-(1)x-5x}\ +5=x(x-1)-5(x-1)=(x-1)(x-5)$$

2.112. Two factors of "+14" whose sum equals "−9" are "−2" and "−7"; thus

$$x^2\underline{-9x}\ +14 = x^2\underline{-2x-7x}\ +14 = x(x-2)-7(x-2) = (x-2)(x-7)$$

2.113. Two factors of "+8" whose sum equals "−6" are "−2" and "−4"; thus,

$$x^2\underline{-6x}\ +8 = x^2\underline{-2x-4x}\ +8 = x(x-2)-4(x-2) = (x-2)(x-4)$$

2.114. If the leading coefficient $a \neq 1$ in ax^2+bx+c, e.g., $a = 2 \neq 1$, then,

Method 1: first, factor the leading coefficient in trinomial as $ax^2+bx+c = a[x^2+(b/a)x+(c/a)]$, then search factors of (c/a) whose sum equals (b/a), i.e.

$$\begin{aligned}
2x^2+3x+1 &= 2\left(x^2+\underline{\frac{3}{2}x}\ +\frac{1}{2}\right)\\
&= \left\{\text{factors of } 1/2 \text{ result in, } 1\times\frac{1}{2}=\frac{1}{2} \quad \text{and} \quad 1+\frac{1}{2}=\frac{3}{2}\right\}\\
&= 2\left(x^2+\underline{x+\frac{1}{2}x}\ +\frac{1}{2}\right) = 2\left[x(x+1)+\frac{1}{2}(x+1)\right]\\
&= 2\left[(x+1)\left(x+\frac{1}{2}\right)\right] = (x+1)(2x+1)
\end{aligned}$$

Method 2: search factors of (ac) whose sum equals b, i.e.

$$\begin{aligned}
2x^2+3x+1 &= \{ac = 2\times 1 = 2 \ \ \therefore\ \ 1\times 2 = 2 \quad \text{and} \quad 1+2=3\}\\
&= 2x^2\underline{+3x}\ +1 = 2x^2\underline{+2x+x}\ +1 = 2x(x+1)+x+1\\
&= (x+1)(2x+1)
\end{aligned}$$

2.115. Search factors of (ac) whose sum equals b, i.e.

$$\begin{aligned}
3x^2\underline{-14x}\ +8 &= \left\{\begin{array}{l}3\times 8 = 24, \quad (-12)\times(-2) = 24 \quad \text{and}\\ (-12)+(-2) = -14\end{array}\right\}\\
&= 3x^2\underline{-12x-2x}\ +8 = 3x(x-4)-2(x-4)\\
&= (x-4)(3x-2)
\end{aligned}$$

2.116. Search factors of (ac) whose sum equals b, i.e.

$$\begin{aligned}
-2x^2\underline{+x}\ +3 &= \{-2\times 3 = -6, \quad (-2)+3 = 1\ \}\\
&= -2x^2\underline{-2x+3x}\ +3 = -2x(x+1)+3(x+1)\\
&= (x+1)(3-2x)
\end{aligned}$$

2.117. $-x^2\underline{+5x}\ -4 = -x^2\underline{+x+4x}\ -4 = -x(x-1)+4(x-1) = (x-1)(4-x)$

2.118. $-6x^2\underline{+5x}\ +4 = -6x^2\underline{-3x+8x}\ +4 = -3x(2x+1)+4(2x+1) = (2x+1)(4-3x)$

2.119. $x^2+(\sqrt{2}+\sqrt{3})x+\underline{\sqrt{6}} = x^2+x\sqrt{2}+x\sqrt{3}+\underline{\sqrt{2\times 3}} = x(x+\sqrt{2})+\sqrt{3}(x+\sqrt{2})$
$= (x+\sqrt{2})(x+\sqrt{3})$

2.120. After writing x^{2n} as $(x^n)^2$, this bi-quadratic polynomial form is converted into quadratic relative to (x^n), then by applying Viète formulas it follows:

$$x^4\underline{-13x^2}\ +36 = (x^2)^2\underline{-4(x^2)-9(x^2)}\ +36 = x^2(x^2-4)-9(x^2-4)$$
$$= (x^2-4)(x^2-9) = (x+2)(x-2)(x+3)(x-3)$$

2.121. After writing x^{2n} as $(x^n)^2$, this bi-quadratic polynomial form is converted into quadratic relative to (x^n), then by applying Viète formulas it follows:

$$x^4\underline{-10x^2}\ +9 = x^4\underline{-x^2-9x^2}\ +9 = x^2(x^2-1)-9(x^2-1) = (x^2-1)(x^2-9)$$
$$= (x+1)(x-1)(x+3)(x-3)$$

2.122. Convert this bi-quadratic polynomial form into quadratic form, as

$$x^6-2x^3+1 = (x^3)^2-2x^3+1 = (x^3-1)^2$$
$$= \left\{(x^3-1)\div(x-1) = x^2+x+1\right\}$$
$$= [(x^2+x+1)(x-1)]^2 = (x^2+x+1)^2(x-1)^2$$
$$= (x+1)(x-1)(x^2+x+1)^2$$

2.123. This is bi-quadratic polynomial; thus

$$x^6\underline{+3x^3}\ +2 = x^6+\underline{x^3+2x^3}\ +2 = x^3(x^3+1)+2(x^3+1)$$
$$= (x^3+1)(x^3+2)\ \{\text{(by long divisions)}\}$$
$$= (x+1)(x^2-x+1)(x+\sqrt[3]{2})(x^2-x\sqrt[3]{2}+\sqrt[3]{4})$$

2.124. This is bi-quadratic polynomial; thus

$$x^6\underline{+9x^3}\ +8 = x^6\underline{+x^3+8x^3}\ +8 = x^3(x^3+1)+8(x^3+1)$$
$$= (x^3+1)(x^3+8) = (x+1)(x^2-x+1)(x^3+8)$$
$$= (x+1)(x^2-x+1)(x+2)(x^2-2x+4)$$

2.125. This is bi-quadratic polynomial; thus

$$2x^{2m+2}-11x^{m+1}+9 = 2\left(x^{m+1}\right)^2-11\left(x^{m+1}\right)+9 = 2\left(x^{m+1}\right)^2-2\left(x^{m+1}\right)-9\left(x^{m+1}\right)+9$$
$$= 2x^{m+1}\left(x^{m+1}-1\right)-9\left(x^{m+1}-1\right) = \left(x^{m+1}-1\right)\left(2x^{m+1}-9\right)$$

2.7 Completing the Square

2.126. Complete the binomial square form as follows:

$$x^2+6x-7=\underbrace{x^2+2(3)x+3^2}_{(x+3)^2}\underbrace{-3^2-7}_{-16}=(x+3)^2-16=0$$

Therefore, the quadratic equation is solved as

$$(x+3)^2=16 \;\;\therefore\;\; x+3=\pm 4 \;\;\therefore\;\; x_{1,2}=-3\pm 4 \;\;\therefore\;\; x_1=1,\;\; x_2=-7$$

2.127. The leading coefficient (i.e., "2") is not equal to one so it can be factored. In addition, the linear term coefficient "5" is a prime number, consequently it is not possible to factor "2" that is necessary for the complete square form. The workaround is to multiply and divide "5" by two, as

$$\begin{aligned}
2x^2-10x-3&=2\left(x^2-5x-\frac{3}{2}\right)=2\left(x^2-2\left(\frac{5}{2}\right)x-\frac{3}{2}\right)\\
&=2\left(\underbrace{x^2-2\left(\frac{5}{2}\right)x+\left(\frac{5}{2}\right)^2}-\left(\frac{5}{2}\right)^2-\frac{3}{2}\right)\\
&=2\left[\left(x-\frac{5}{2}\right)^2-\frac{25}{4}-\frac{3}{2}\right]=2\left[\left(x-\frac{5}{2}\right)^2-\frac{31}{4}\right]=0 \;\;\therefore\;\; \left(x-\frac{5}{2}\right)^2-\frac{31}{4}=0\\
\therefore\;\; \left(x-\frac{5}{2}\right)^2&=\frac{31}{4} \;\;\therefore\;\; x-\frac{5}{2}=\pm\sqrt{\frac{31}{4}} \;\;\therefore\;\; x_{1,2}=\frac{5}{2}\pm\sqrt{\frac{31}{4}}=\frac{5\pm\sqrt{31}}{2}
\end{aligned}$$

2.128. Factor the leading coefficient "-1" as

$$\begin{aligned}
-x^2-6x+7=-[x^2+2(3)x+3^2-3^2-7]&=-[(x+3)^2-16]=0\\
\therefore\;\; (x+3)^2-16&=0\\
\therefore\;\; x+3&=\pm\sqrt{16}\\
\therefore\;\; x_{1,2}&=-3\pm 4\\
\therefore\;\; x_1=1,\;\; x_2&=-7
\end{aligned}$$

Note however, that although the roots of $(-x^2-6x+7)$ and (x^2+6x-7) are same, these two quadratic polynomials are not the equal.

2.129. For the sake of exercising,
Method 1: Force the completing square form as

$$x^2+3x=x^2+2\left(\frac{3}{2}\right)x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2=\left(x+\frac{3}{2}\right)^2-\frac{9}{4}=0$$

$$\therefore\; x+\frac{3}{2}=\pm\frac{3}{2} \;\;\therefore\;\; x_{1,2}=-\frac{3}{2}\pm\frac{3}{2} \;\;\therefore\;\; x_1=0,\;\; x_2=-3$$

Method 2: Of course, in this case factorization is faster

$$x^2 + 3x = x(x-3) \;\; \therefore \;\; x_1 = 0, \;\; x_2 = -3$$

2.130. Factorize the leading coefficient, as well as $6 = 2 \times 3$, so that

$$\begin{aligned} 2x^2 + 6x + 2 &= 2\left[x^2 + 2\left(\frac{3}{2}\right)x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 1\right] \\ &= 2\left[\left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + 1\right] = 0 \\ \therefore \; x + \frac{3}{2} &= \pm\sqrt{\frac{5}{4}} \;\; \therefore \;\; x_{1,2} = \frac{-3 \pm \sqrt{5}}{2} \end{aligned}$$

2.131. $x^2 - x - 6 = 0 \;\; \therefore \;\; \left(x - \frac{1}{2}\right)^2 - \frac{25}{4} = 0 \;\; \therefore \;\; x_1 = -2, \;\; x_2 = 3$

2.8 Factor Theorem

2.132. Given $Q_3(x) = 1x^3 - 1$, the only two factors of $a_3\, a_0 = 1 \times (-1) = -1$ product are "+1" and "−1." That is because "$-1 \times 1 = -1$" and "$1 \times (-1) = -1$." Possible roots of $Q_3(x)$ may be one of these two factors; thus, first step is to verify if $Q_3(\pm 1) = 0$, as

$$\begin{aligned} Q_3(-1) &= (-1)^3 - 1 = -2 \;\; \therefore \;\; Q(-1) \neq 0 \\ Q_3(1) &= (1)^3 - 1 = 0 \;\; \therefore \;\; x_1 = 1 \end{aligned}$$

That is to say, because $Q(1) = 0$, it must be that $x_1 = 1$ is one of its roots; by consequence, $(x-1)$ is one of $Q_3(x)$ factors, and furthermore $Q_3(x)$ polynomial can be factorized as

$$Q_3(x) = (x-1)(ax^2 + bx + c) \;\; \Rightarrow Q_3(x) \div (x-1) = ax^2 + bx + c$$

where, $(ax^2 + bx + c)$ must be second-order polynomial that is left after the factorization, i.e.,

$$ax^2 + bx + c = Q_3(x) \div (x - x_1) \;\; \Rightarrow ax^2 + bx + c = (x^3 - 1) \div (x-1)$$

The long polynomial division results in

$$\begin{array}{r} (x^3 - 1) \div (x-1) = x^2 + x + 1 \\ \underline{-(x^3 - x^2)} \\ \quad x^2 - 1 \\ \quad \underline{-(x^2 - x)} \\ \qquad\quad x - 1 \end{array}$$

$$
\begin{aligned}
&\underline{-\,(x-1)}\\
&\underline{\quad\;=0}
\end{aligned}
$$

Therefore, $Q_3(x) = x^3 - 1 = (x-1)(x^2+x+1)$. The quadratic polynomial factor may be itself further factorized by multiple methods. For example, its discriminant is negative, therefore the last two roots of Q_3 are complex,

$$
\begin{aligned}
\Delta &= b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = -3 < 0\\
&\therefore\\
x_{2,3} &= \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-1 \pm i\sqrt{3}}{2}\\
&\therefore\\
Q(x) &= x^3 - 1 = (x-1)(x^2+x+1)\\
&= (x-1)\left(x + \frac{1+i\sqrt{3}}{2}\right)\left(x + \frac{1-i\sqrt{3}}{2}\right)
\end{aligned}
$$

Note that this procedure for factorizing higher-order polynomials is quite general, and it illustrates the use of basic theorems of algebra.

2.133. An important polynomial form is known as the *difference of two cubes*. Given $P(x) = x^3 - y^3$ note that P.2.132 may be considered as special case where $y = 1$. Then, similarly, by inspection of $P(x)$ it may be concluded that one of its roots must be $x_1 = y$, that is because $P(y) = 0$. Therefore, $P(x)$ is divisible by $(x - y)$.

The long polynomial division results in

$$
\begin{aligned}
&(x^3 + 0x^2y + 0xy^2 - y^3) \div (x-y) = x^2 + xy + y^2\\
&\underline{x^3 - x^2y}\\
&\quad x^2y + 0xy^2\\
&\quad \underline{x^2y - xy^2}\\
&\qquad\quad xy^2 - y^3\\
&\qquad\quad \underline{xy^2 - y^3}\\
&\qquad\quad\;\; \underline{=0}
\end{aligned}
$$

Therefore, $(x^3 - y^3) = (x-y)(x^2+xy+y^2)$, and it is usually listed among other basic polynomial identities, e.g., difference of squares, square of a binomial, etc..

2.134. Given $R_3(x) = 1x^3 + 1$, the only two factors of $a_3\, a_0 = 1 \times 1 = 1$ product are "+1" and "−1." That is because"$(-1) \times (-1) = 1$" and "$1 \times 1 = 1$." Possible roots of $R_3(x)$ may be one of these two factors; thus, first step is to verify if $R_3(\pm 1) = 0$, as

$$R_3(-1) = (-1)^3 + 1 = 0 \;\; \therefore \;\; x_1 = -1$$
$$R_3(1) = (1)^3 + 1 = 2 \;\; \therefore \;\; R_3(1) \neq 0$$

That is to say, because $R(-1) = 0$, it must be that $x_1 = -1$ is one of its roots; by consequence, $\big(x - (-1)\big)$ is one of $R_3(x)$ factors, and furthermore $R_3(x)$ polynomial can be factorized as

$$R_3(x) = (x+1)(ax^2 + bx + c) \;\; \Rightarrow R_3(x) \div (x+1) = ax^2 + bx + c$$

where, $(ax^2 + bx + c)$ must be second-order polynomial that is left after the factorization, i.e.,

$$ax^2 + bx + c = R_3(x) \div (x - x_1) \;\; \Rightarrow ax^2 + bx + c = (x^3 + 1) \div (x+1)$$

The long polynomial division results in

$$\begin{aligned}
&(x^3 + 1) \div (x+1) = x^2 - x + 1 \\
&-\underline{(x^3 + x^2)} \\
&\quad -x^2 + 1 \\
&\quad -\underline{(-x^2 - x)} \\
&\qquad\quad x + 1 \\
&\qquad -\underline{(x+1)} \\
&\qquad\quad = 0
\end{aligned}$$

Therefore, $R_3(x) = x^3 + 1 = (x+1)(x^2 - x + 1)$. The quadratic polynomial factor may be itself further factorized by multiple methods. For example, its discriminant is negative, therefore the last two roots of R_3 are complex,

$$\Delta = b^2 - 4ac = (-1)^2 - 4 \times 1 \times 1 = -3 < 0$$
$$\therefore$$
$$x_{2,3} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{1 \pm i\sqrt{3}}{2}$$
$$\therefore$$
$$\begin{aligned}
R(x) &= x^3 + 1 = (x+1)(x^2 - x + 1) \\
&= (x+1)\left(x - \frac{1 + i\sqrt{3}}{2}\right)\left(x - \frac{1 - i\sqrt{3}}{2}\right)
\end{aligned}$$

2.135. Given $R_3(x) = 1x^3 + x + 2$, factors of $a_3\, a_0 = 1 \times 2 = 1$ product are "$+1, +2$" and "$-1, -2$." Possible roots of $R_3(x)$ may be one of these factors. Thus, by inspection, $R_3(-1) = 0$, as $(-1^3) + (-1) + 2 = 0$. The long polynomial division results in

$$
\begin{aligned}
&(x^3 + x + 2) \div (x + 1) = x^2 - x + 2 \\
&\underline{-(x^3 + x^2)} \\
&\quad -x^2 + x \\
&\quad \underline{-(-x^2 - x)} \\
&\qquad\quad 2x + 2 \\
&\qquad\quad \underline{-(2x + 2)} \\
&\qquad\quad\ \ = 0
\end{aligned}
$$

Therefore, $(x^3 + x + 2) = (x + 1)(x^2 - x + 2)$. The quadratic term has two complex zeros.

2.136. Given $P(x) = x^3 + y^3$ note that P.2.134 may be considered as special case where $y = 1$. Then, similarly, by inspection of $P(x)$ it may be concluded that one of its roots must be $x_1 = -y$, that is because $P(-y) = 0$. Therefore, $P(x)$ is divisible by $\big(x - (-y)\big)$.

The long polynomial division results in

$$
\begin{aligned}
&(x^3 + 0x^2y + 0xy^2 + y^3) \div (x + y) = x^2 - xy + y^2 \\
&\underline{x^3 + x^2y} \\
&\quad -x^2y + 0xy^2 \\
&\quad \underline{-x^2y - xy^2} \\
&\qquad\quad xy^2 + y^3 \\
&\qquad\quad \underline{xy^2 + y^3} \\
&\qquad\quad\ \ \underline{= 0}
\end{aligned}
$$

Therefore, $(x^3 + y^3) = (x + y)(x^2 - xy + y^2)$, and it is usually listed among other basic polynomial identities, e.g., difference of squares, square of a binomial, etc..

2.137. Factorization of higher-order polynomials usually requires multiple steps and techniques. Consequently, there is no unique path to the solution; sometimes, the solutions are elegant, as

$$
\begin{aligned}
x^4 + 4 &= \underline{x^4 + 4x^2 + 4}\ -4x^2 = (x^2 + 2)^2 - (2x)^2 = \big\{a^2 - b^2 = (a + b)(a - b)\big\} \\
&= (x^2 + 2x + 2)(x^2 - 2x + 2)
\end{aligned}
$$

which is to say that all four zeros of $(x^4 + 4)$ are complex.

2.138. Similarly to P.2.137, it is possible to take advantage of difference of squares identity, as

$$
x^4 + x^2 + 1 = \underline{x^4 + 2x^2 + 1}\ -x^2 = (x^2 + 1)^2 - (x)^2 = (x^2 - x + 1)(x^2 + x + 1)
$$

thus, all four zeros of $(x^4 + x^2 + 1)$ are complex.

2.139. As some prerequisite work is already done in the previous examples,

$$\begin{aligned} x^5+x+1 &= x^5 - x^2 + x^2 + x + 1 = x^2(x^3-1) + x^2 + x + 1 \\ &= \{x^3 - 1 = (x-1)(x^2+x+1) \text{ see A.2.132 }\} \\ &= x^2(x-1)(x^2+x+1) + (x^2+x+1) \\ &= (x^2+x+1)\big(x^2(x-1)+1\big) \\ &= (x^2+x+1)(x^3-x^2+1) \end{aligned}$$

Where, quadratic polynomial (x^2+x+1) has two complex zeros, and cubic polynomial (x^3-x^2+1) has one real and two complex zeros (see chapter on function analysis).

2.140. As some prerequisite work is already done in the previous examples,

$$\begin{aligned} (x+y+z)^3 - x^3 - y^3 - z^3 &= (x+y+z)^3 - x^3 - (y^3+z^3) \\ &= \left\{ \begin{matrix} a^3-b^3 & = (a-b)(a^2+ab+b^2) \text{ see P.2.133} \\ a^3+b^3 & = (a+b)(a^2-ab+b^2) \end{matrix} \right\} \\ &= (\cancel{x}+y+z-\cancel{x})\big((x+y+z)^2 + (x+y+z)\,x + x^2\big) - (y+z)(y^2-yz+z^2) \\ &= (y+z)\big((x+y+z)^2 + (x+y+z)\,x + x^2 - y^2 + yz - z^2\big) \\ &= (y+z)\big(2xy + 2xz + x^2 \cancel{+y^2} + 2yz \cancel{+z^2} + x^2 + xy + zx + x^2 \cancel{-y^2} + yz \cancel{-z^2}\big) \\ &= (y+z)\left(3xy + 3xz + 3yz + 3x^2\right) = 3(y+z)\left(xy + xz + yz + x^2\right) \\ &= 3(y+z)\big(x(x+y) + z(x+y)\big) = 3(y+z)(x+y)(x+z) \end{aligned}$$

2.141. Given $P(x) = x^3 + x^2 - 4x - 4$, factors of "$-4 \times 1 = -4$" are "$\{\pm 1, \pm 4, \pm 2\}$." That is because:

$$(-1) \times 4 = -4, \quad 1 \times (-4) = -4, \quad (-2) \times 2 = -4$$

Thus, $P(x)$ calculated for each factor is

$$\begin{aligned} P(-1) &= ((-1)^3 + (-1)^2 - 4(-1) - 4) = 0 \;\; \therefore \;\; x_1 = -1 \\ P(1) &= ((1)^3 + (1)^2 - 4(1) - 4) = -6 \;\; \therefore \;\; P(1) \neq 0 \\ P(-2) &= ((-2)^3 + (-2)^2 - 4(-2) - 4) = 0 \;\; \therefore \;\; x_2 = -2 \\ P(2) &= ((2)^3 + (2)^2 - 4(2) - 4) = 0 \;\; \therefore \;\; x_3 = 2 \\ P(-4) &= ((-4)^3 + (-4)^2 - 4(-4) - 4) = -36 \;\; \therefore \;\; P(-4) \neq 0 \\ P(4) &= ((4)^3 + (4)^2 - 4(4) - 4) = 60 \;\; \therefore \;\; P(4) \neq 0 \end{aligned}$$

Therefore,

$$P(x) = x^3 + x^2 - 4x - 4 = (x - x_1)(x - x_2)(x - x_3) = (x + 1)(x + 2)(x - 2)$$

As it happen, all three roots are real.

2.142. By algebraic transformations,

$$\begin{aligned}
\left(x^2 + x + 1\right)\left(x^3 + x^2 + 1\right) - 1 &= (x^2 + 1 \ + x\ )(x^3 \ + x^2 + 1\ ) - 1 \\
&\quad \{\text{first, expand by } (x^2 + 1) \text{ left to right as,}\} \\
&= \left(x^2 + 1\right) x^3 + \left(x^2 + 1\right)\left(x^2 + 1\right) \\
&\quad \{\text{then, expand by } x \text{ as,}\} \\
&\quad + x\, x^3 \ + x\left(x^2 + 1\right) \ -1 \\
&\quad \{x^4 - 1, \quad \text{use difference of squares identity}\} \\
&= \left(x^2 + 1\right) x^3 + \left(x^2 + 1\right)^2 + \left(x^2 + 1\right) x + \left(x^2 - 1\right)\left(x^2 + 1\right) \\
&= \left(x^2 + 1\right)\left(x^3 + \left(x^2 \cancel{+1}\right) + x + \left(x^2 \cancel{-1}\right)\right) \\
&= \left(x^2 + 1\right)\left(x^3 + 2x^2 + x\right) = x\left(x^2 + 1\right)\left(x^2 + 2x + 1\right) \\
&= x\left(x^2 + 1\right)(x + 1)^2
\end{aligned}$$

2.143. In many occasions, the change of variables technique is best way to temporarily simplify complicated forms. Along the way, one can take advantage of any other applicable identity as

$$\begin{aligned}
(x + 1)(x + 3)(x + 5)(x + 7) + 15 &= (x + 1)(x + 7)\ (x + 3)(x + 5) + 15 \\
&= (x^2 + 8x + 7)(x^2 + 8x + 15) + 15
\end{aligned}$$

$$\begin{aligned}
&= (x^2 + 8x + 7\ )(x^2 + 8x + 7\ + 8) + 15 \\
&= \{\text{substitution: } x^2 + 8x + 7 = t\} \\
&= t(t + 8) + 15 = t^2 + 8t + 15 \\
&= t^2 + 3t + 5t + 15 = t(t + 3) + 5(t + 3) = (t + 3)(t + 5) \\
&= \{\text{back substitution: } t = x^2 + 8x + 7\} \\
&= \left(x^2 + 8x + 7 + 3\right)\left(x^2 + 8x + 7 + 5\right) \\
&= \left(x^2 + 8x + 10\right)\left(x^2 + 8x + 12\right) \\
&= \left(x^2 + 2(4)x \ +4^2 - 4^2 + 10\right)\left(x^2 + 2x + 6x + 12\right) \\
&= \left((x + 4)^2 - 6\right)(x + 2)(x + 6) \\
&= \{a^2 - b^2 = (a + b)(a - b)\} \\
&= (x + 4 - \sqrt{6})(x + 4 + \sqrt{6})(x + 2)(x + 6)
\end{aligned}$$

2.144. Two polynomials are equal if each of their corresponding monomials are equal. First, product of $(x+1)$ and parametric quadratic polynomial is

$$\begin{aligned} Q(x) &= (x+1)(ax^2+bx+c) = ax^3+bx^2+cx+ax^2+bx+c \\ &= ax^3+(a+b)x^2+(b+c)x+c \quad \therefore \\ \text{if,} \quad Q(x) &= P(x) \quad \Rightarrow \quad ax^3+(a+b)x^2+(b+c)x+c = x^3-2x^2+3 \end{aligned}$$

x^3 terms: $\therefore \quad a = 1$

x^0 terms: $\therefore \quad c = 3$

x^2 terms: $a+b=-2 \quad \therefore \quad 1+b=-2 \quad \therefore \quad b=-3$

x^1 terms: $b+c=0 \quad \therefore \quad -3+3=0$ ✓

In conclusion, $P(x) = Q(x) \quad \therefore \quad x^3-2x^2+3 = (x+1)(x^2-3x+3)$

2.145. Similarly to A.2.144,

$$\begin{aligned} Q(x) &= (x-2)(ax^2+bx+c) = ax^3+bx^2+cx-2ax^2-2bx-2c \quad \therefore \\ \text{if,} \quad Q(x) &= P(x) \quad \Rightarrow \quad ax^3+(b-2a)x^2+(c-2b)x-2c = 2x^3-x^2+x+4 \end{aligned}$$

$\therefore$

x^3 terms: $\therefore \quad a = 2$

x^0 terms: $-2c = 4 \quad \therefore \quad c = -2$

x^2 terms: $b-2a=-1 \quad \therefore \quad b-4=-1 \quad \therefore \quad b=3$

x^1 terms: $c-2b=1 \quad \therefore \quad -2-6 \neq 1$ ✗

In conclusion, there is no $Q(x)$ that satisfies this $P(x) = Q(x)$ equation.

2.146. Given polynomial $P(x) = (x^4-2x^3+3x^2-4x+1)$ and divisor $(x-1)$, i.e., $n=1$,

$$r = P(1) = (1)^4-2(1)^3+3(1)^2-4(1)+1) = -1$$

Therefore $n=1$ is not the root of $P(x)$, consequently $(x-1)$ is not one of $P(x)$ factors because reminder of this division is

$$R = -\frac{1}{x-1}$$

2.147. Given polynomial $P(x) = (x^3 + 1)$ and divisor $(x + 1) = (x - (-1))$, i.e., $n = -1$,

$$r = P(-1) = (-1)^3 + 1 = 0$$

Therefore $n = -1$ is one of the roots of $P(x)$, consequently reminder $R = 0$, in other words, $P(x)$ is divisible by $(x + 1)$. Indeed,

$$\frac{x^3 + 1}{x + 1} = x^2 - x + 1$$

2.148. Given polynomial $P(x) = \left(x^{1965} - 256x^{1961} + 1\right)$ and its divisor $Q(x) = x^2 - 4x$, it means that if $P(x)$ is divisible by $Q(x)$ it must be divisible by all factors of $Q(x)$ as well, where

$$Q(x) = x^2 - 4x = x(x - 4) = (x - 0)(x - 4)$$

It is necessary therefore to calculate reminders for both $(x - 0)$ and $(x - 4)$ factors as

$$\begin{aligned} r(0) = P(0) &= \cancel{(0)^{1965}} \to 0 - 256\cancel{(0)^{1961}} \to 0 + 1 = 1 \\ r(4) = P(4) &= (4)^{1965} - 256(4)^{1961} + 1 = 4^{1965} - 4^4\,(4)^{1961} + 1 \\ &= \cancel{4^{1965}} - \cancel{4^{1965}} + 1 = 1 \end{aligned}$$

In conclusion, both reminders are equal $r = 1$; therefore, reminder of this division is

$$R = \frac{1}{x^2 - 4x} \quad \Rightarrow \frac{P(x)}{Q(x)} = N_{1963}(x) + \frac{1}{x^2 - 4x}$$

where $N_{1963}(x)$ is a 1963th-order polynomial. This is because 1965th-order polynomial is divided by second-order polynomial; the resulting polynomial must be $1965 - 2 = 1963$rd order.

2.149. Given polynomial $P(x) = (x^{2023} - 2x^{2022} - 1)$ and its divisor $Q(x) = (x^2 - 3x + 2)$, it means that if $P(x)$ is divisible by $Q(x)$, it must be divisible by all factors of $Q(x)$ as well, where

$$Q(x) = x^2 - 3x + 2 = x^2 - x - 2x + 2 = x(x - 1) - 2(x - 1) = (x - 1)(x - 2)$$

It is necessary to calculate reminders at two points, i.e., for both $(x - 1)$ and $(x - 2)$ factors as

$$\begin{aligned} r(1) = P(1) &= \cancel{(1)^{2023}} \to 1 - 2\cancel{(1)^{2022}} \to 1 - 1 = -2 \\ r(2) = P(2) &= (2)^{2023} - 2(2)^{2022} - 1 = \cancel{2^{2023}} - \cancel{2^{2023}} - 1 = -1 \end{aligned}$$

There are two different values of r, and there are two points in consideration, $x = 1$ and $x = 2$; the conclusion is that reminder must be in the form of first-order polynomial $r(x) = ax + b$, so that

$$\left.\begin{aligned} \underline{x = 1} &: \; r(1) = -2 = a(1) + b \\ \underline{x = 2} &: \; r(2) = -1 = a(2) + b \end{aligned}\right\} \quad \therefore \quad \left.\begin{aligned} a + b &= -2 \\ 2a + b &= -1 \end{aligned}\right\} \quad \therefore \;\; a = 1, \;\; b = -3$$

In conclusion, the reminder in division $P(x)/Q(x)$ is $r(x) = ax + b = x - 3$, so that

$$R = \frac{x-3}{x^2-3x+2} \quad \Rightarrow \frac{P(x)}{Q(x)} = N_{2021}(x) + \frac{x-3}{x^2-3x+2}$$

where $N_{2021}(x)$ is a 2021th-order polynomial. This is because 2023th-order polynomial is divided by second-order polynomial; the resulting polynomial must be $2023 - 2 = 2021^{st}$ order.

2.150. Given parametrized polynomial $P(x, k) = 4x^5 + k\,x^4 + 8x^3 + 5x^2 + 3x + 2$ and its divisor $Q(x) = x + 2$, i.e., $n = -2$, for $P(x)$ to be divisible by $(x + 2)$, it is necessary that $r = P(-2) = 0$. Therefore

$$\begin{aligned} r = P(-2) &= 4(-2)^5 + k(-2)^4 + 8(-2)^3 + 5(-2)^2 + 3(-2) + 2 \\ &= -128 + 16k - 64 + 20 - 6 + 2 = 0 \quad \therefore \quad 16k - 176 = 0 \\ \therefore \quad k &= \frac{176}{16} = 11 \end{aligned}$$

Indeed,

$$\frac{4x^5 + 11x^4 + 8x^3 + 5x^2 + 3x + 2}{x + 2} = 4x^4 + 3x^3 + 2x^2 + x + 1$$

2.151. Given parametrized polynomial $P(x, k) = x^3 - 3k\,x + 4(k^2 + 1)x - (k^3 + 5)$ and its divisor $Q(x) = x - 1$, i.e., $n = 1$, for $P(x, k)$ to be divisible by $(x - 1)$, it is necessary that $r = P(1) = 0$. Therefore

$$\begin{aligned} r = P(1) &= (1)^3 - 3k(1) + 4(k^2 + 1)(1) - (k^3 + 5) \\ &= 1 - 3k + 4k^2 + 4 - k^3 - 5 \\ &= -k^3 + 4k^2 - 3k = -k(k^2 - 4k + 3) = -k(k^2 - k - 3k + 3) \\ &= -k\big(k(k-1) - 3(k-1)\big) = -k(k-1)(k-3) = 0 \end{aligned}$$

Therefore, there are three values of k that produce $r = 0$, i.e., $k = 0, 1, 3$. Indeed,

$$\begin{aligned} k = 0: \quad & \frac{P(x,k)}{x-1} = \frac{x^3 + 4x - 5}{x-1} = x^2 + x + 5 \\ k = 1: \quad & \frac{P(x,k)}{x-1} = \frac{x^3 + 5x - 6}{x-1} = x^2 + x + 6 \\ k = 3: \quad & \frac{P(x,k)}{x-1} = \frac{x^3 + 31x - 32}{x-1} = x^2 + x + 32 \end{aligned}$$

2.9 Partial Fraction Decomposition

2.152. Given

$$\frac{P(x)}{Q(x)} = \frac{x}{(x+1)(x-4)}$$

denominator $Q(x)$ has two unique zeros at $x = -1$ and $x = 4$, thus

$$\frac{P(x)}{Q(x)} = \frac{x}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4} = \frac{A(x-4)+B(x+1)}{(x+1)(x-4)} = \frac{Ax-4A+Bx+B}{(x+1)(x-4)}$$
$$= \frac{(A+B)x+B-4A}{(x+1)(x-4)}$$

Denominator $Q(x)$ did not change; therefore, numerator $P(x)$ must not change before and after decomposition, that is,

$$x \equiv (A+B)x + B - 4A$$
$$\therefore$$
$$\underline{\text{term } x^1} \; : \; 1 = A + B \;\; \therefore \;\; A = 1 - B$$
$$\underline{\text{term } x^0} \; : \; 0 = B - 4A \;\; \therefore \;\; B = 4A$$
$$\therefore$$
$$B = 4(1-B), \;\; \therefore \;\; B = \frac{4}{5} \;\; \therefore \;\; A = 1 - \frac{4}{5} = \frac{1}{5}$$

Finally,

$$\frac{P(x)}{Q(x)} = \frac{x}{(x+1)(x-4)} = \frac{1}{5(x+1)} + \frac{4}{5(x-4)}$$

2.153. Given

$$\frac{P(x)}{Q(x)} = \frac{x+2}{x^3-2x^2} = \frac{x+2}{x^2(x-2)}$$

denominator $Q(x)$ has one unique zero at $x = 2$ and one double zero $x = 0$, thus

$$\frac{P(x)}{Q(x)} = \frac{x+2}{x^3-2x^2} = \frac{x+2}{x^2(x-2)} = \left\{Q(x) = x^2(x-2) \;\; \therefore \;\; x_{1,2} = 0, \;\; x_3 = 2\right\}$$
$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2)+B(x-2)+Cx^2}{x^2(x-2)}$$
$$= \frac{Ax^2-2Ax+Bx-2B+Cx^2}{x^2(x-2)} = \frac{(A+C)x^2+(B-2A)x-2B}{x^2(x-2)}$$

Denominator $Q(x)$ did not change; therefore, numerator $P(x)$ must not change before and after decomposition, that is,

$$x+2 \equiv (A+C)x^2 + (B-2A)x - 2B$$

$$\therefore$$

$$\underline{\text{term } x^2} \;:\; 0 = A + C \;\;\therefore\;\; A = -C$$

$$\underline{\text{term } x^1} \;:\; 1 = B - 2A$$

$$\underline{\text{term } x^0} \;:\; 2 = -2B \;\;\therefore\;\; B = -1$$

$$\therefore$$

$$B = -1, \;\;\therefore\;\; 1 = (-1) - 2A \;\;\therefore\;\; A = -1 \;\;\therefore\;\; C = 1$$

Finally,

$$\frac{P(x)}{Q(x)} = \frac{x+2}{x^3 - 2x^2} = \frac{x+2}{x^2(x-2)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-2}$$

2.154. Given

$$\frac{P(x)}{Q(x)} = \frac{2x^2}{x^4 - 1}$$

denominator $Q(x)$ has two unique real zeros at $x = 1$ and $x = -1$, as well as two complex zeros, as

$$\begin{aligned}
\frac{P(x)}{Q(x)} &= \frac{2x^2}{x^4-1} = \left\{a^2 - b^2 = (a-b)(a+b)\right\} = \frac{2x^2}{(x^2-1)(x^2+1)} \\
&= \frac{2x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \\
&= \frac{A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)}{(x-1)(x+1)(x^2+1)} \\
&= \frac{(A+B+C)x^3 + (A-B+D)x^2 + (A+B-C)x + A - B - D}{(x-1)(x+1)(x^2+1)}
\end{aligned}$$

Denominator $Q(x)$ did not change; therefore, numerator $P(x)$ must not change before and after decomposition, that is,

$$2x^2 \equiv (A+B+C)x^3 + (A-B+D)x^2 + (A+B-C)x + (A-B-D)$$

$$\therefore$$

$$\underline{\text{term } x^3} \;:\; 0 = A + B + C$$

$$\underline{\text{term } x^2} \;:\; 2 = A - B + D$$

$$\underline{\text{term } x^1} \;:\; 0 = A + B - C$$

$$\underline{\text{term } x^0} \;:\; 0 = A - B - D$$

The above system of linear equations may be solved, for example, by the reduction method. From first equation $0 = A + B + C \;\;\therefore\;\; B = -A - C$, then

$$2 = A - B + D \;\therefore\; 2 = A - (-A - C) + D = 2A + C + D$$

$$0 = A + B - C \;\therefore\; 0 = A + (-A - C) - C = -2C \;\therefore\; \underline{C = 0}$$

$$\underline{0 = A - B - D \;\therefore\; 0 = A - (-A - C) - D = 2A - \cancel{C} - D = 2A - D \;\therefore\; 2A = D}$$

$$2 = 2A + \cancel{C} + D \;\therefore\; D = 2 - 2A$$

$$2A = D \;\therefore\; 2A = 2 - 2A \;\therefore\; \underline{A = \frac{1}{2}} \;\therefore\; \underline{D = 1}$$

$$B = -A - \cancel{C} \;\therefore\; \underline{B = -\frac{1}{2}}$$

Finally,

$$\frac{P(x)}{Q(x)} = \frac{2x^2}{x^4 - 1} = \frac{2x^2}{(x-1)(x+1)(x^2+1)} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)} + \frac{1}{x^2+1}$$

2.155. Given,

$$\frac{P_k(x)}{Q_m(x)} = \frac{1}{x^3 - 1}$$

denominator $Q(x)$ has one unique real zero $x = 1$ and two complex zeros, thus

$$\begin{aligned}\frac{P(x)}{Q(x)} = \frac{1}{x^3-1} &= \{\text{see A.2.132}\} = \frac{1}{(x-1)(x^2+x+1)} \\ &= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} = \frac{A(x^2+x+1) + (Bx+C)(x-1)}{(x-1)(x^2+x+1)} \\ &= \frac{(A+B)x^2 + (A-B+C)x + A - C}{(x-1)(x^2+x+1)}\end{aligned}$$

Denominator $Q(x)$ did not change; therefore, numerator $P(x)$ must not change before and after decomposition, that is,

$$1 \equiv (A+B)x^2 + (A-B+C)x + A - C$$

$$\therefore$$

$$\underline{\text{term } x^2} \;:\; 0 = A + B \;\therefore\; A = -B$$

$$\underline{\text{term } x^1} \;:\; 0 = A - B + C \;\therefore\; 0 = (-B) - B + C \;\therefore\; C = 2B$$

$$\underline{\text{term } x^0} \;:\; 1 = A - C \;\therefore\; 1 = (-B) - (2B) \;\therefore\; \underline{B = -\frac{1}{3}}$$

$$\underline{C = -\frac{2}{3}} \;\therefore\; \underline{A = \frac{1}{3}}$$

Finally,

$$\frac{P(x)}{Q(x)} = \frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)}$$

2.156. Given

$$\frac{P(x)}{Q(x)} = \frac{x^3+x-1}{(x^2+2)^2}$$

denominator $Q(x)$ has multiple complex pair of zeros, thus

$$\frac{P(x)}{Q(x)} = \frac{x^3+x-1}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} = \frac{(Ax+B)(x^2+2)+Cx+D}{(x^2+2)^2}$$
$$= \frac{Ax^3+Bx^2+(C+2A)x+2B+D}{(x^2+2)^2}$$

Denominator $Q(x)$ did not change; therefore, numerator $P(x)$ must not change before and after decomposition, that is,

$$x^3+x-1 \equiv Ax^3+Bx^2+(C+2A)x+2B+D$$
$$\therefore$$
$$\underline{\text{term } x^3} \ : \ \underline{A=1}$$
$$\underline{\text{term } x^2} \ : \ \underline{B=0}$$
$$\underline{\text{term } x^1} \ : \ 1 = C+2A \ \ \therefore \ \ \underline{C=-1}$$
$$\underline{\text{term } x^0} \ : \ -1 = 2\overset{0}{\cancel{B}} + D \ \ \therefore \ \ \underline{D=-1}$$

Finally,

$$\frac{P(x)}{Q(x)} = \frac{x^3+x-1}{(x^2+2)^2} = \frac{x}{x^2+2} - \frac{x+1}{(x^2+2)^2}$$

2.157. Given

$$\frac{P(x)}{Q(x)} = \frac{4}{x^4+1}$$

numerator $Q(x)$ has two complex pairs of zeros. Complete the squares to enforce bi-quadratic form, and then use difference of squares identities to rearrange resulting fourth-order polynomial.

$$\frac{P(x)}{Q(x)} = \frac{4}{x^4+1} = \frac{4}{\underbrace{x^4+2x^2+1}\ -2x^2} = \frac{4}{(x^2+1)^2-\left(\sqrt{2}\,x\right)^2}$$
$$= \frac{4}{(x^2+1-\sqrt{2}\,x)(x^2+1+\sqrt{2}\,x)} = \frac{Ax+B}{x^2-\sqrt{2}\,x+1} + \frac{Cx+D}{x^2+\sqrt{2}\,x+1}$$

$$= \frac{(Ax+B)(x^2+\sqrt{2}\,x+1)+(Cx+D)(x^2-\sqrt{2}\,x+1)}{(x^2-\sqrt{2}\,x+1)(x^2+\sqrt{2}\,x+1)} =$$

$$\frac{(A+C)x^3+(\sqrt{2}A+B-\sqrt{2}C+D)x^2+(A+C+\sqrt{2}B-\sqrt{2}D)x+B+D}{(x^2-\sqrt{2}\,x+1)(x^2+\sqrt{2}\,x+1)}$$

Denominator $Q(x)$ did not change; therefore, numerator $P(x)$ must not change before and after decomposition, that is,

$$4 \equiv (A+C)x^3+(\sqrt{2}A+B-\sqrt{2}C+D)x^2+(A+C+\sqrt{2}B-\sqrt{2}D)x+B+D$$

$\therefore$

$$\underline{\text{term } x^3} \;:\; 0 = A+C \;\;\therefore\;\; A=-C$$

$$\underline{\text{term } x^2} \;:\; 0 = \sqrt{2}A+B-\sqrt{2}C+D \;\;\therefore\;\; 0=-2\sqrt{2}C+B+D$$

$$\underline{\text{term } x^1} \;:\; 0 = A+C+\sqrt{2}B-\sqrt{2}D \;\;\therefore\;\; 0=+\sqrt{2}B-\sqrt{2}D \;\;\therefore\;\; B=D$$

$$\underline{\text{term } x^0} \;:\; 4 = B+D \;\;\therefore\;\; 4=D+D \;\;\therefore\;\; \underline{D=2} \;\;\therefore\;\; \underline{B=2}$$

$$\therefore\;\; 0=-2\sqrt{2}C+4 \;\;\therefore\;\; \underline{C=\sqrt{2}} \;\;\therefore\;\; \underline{A=-\sqrt{2}}$$

Finally,

$$\frac{P(x)}{Q(x)} = \frac{4}{x^4+1} = \frac{-\sqrt{2}x+2}{x^2-\sqrt{2}\,x+1} + \frac{\sqrt{2}x+2}{x^2+\sqrt{2}\,x+1}$$

2.158. Given rational form

$$\frac{x^3+x^2-16x+16}{x^2-4x+3}$$

note that numerator polynomial is of higher order than the denominator. For that reason, first use the long division and then partial fraction decomposition, as

$$(x^3+x^2-16x+16) \div (x^2-4x+3) = x+5+\frac{x+1}{x^2-4x+3}$$

$$\begin{array}{ll} & x^3+x^2-16x+16 \\ (-) & \underline{x^3-4x^2+3x} \\ & 5x^2-19x+16 \\ (-) & \underline{5x^2-20x+15} \\ & x+1 \end{array}$$

Denominator $Q(x)$ has two unique real zeros of $x=1$ and $x=3$, thus

$$\frac{P(x)}{Q(x)} = \frac{x+1}{x^2-4x+3} = \frac{x+1}{(x-1)(x-3)} = \frac{A}{x-1}+\frac{B}{x-3}$$

$$= \frac{A(x-3)+B(x-1)}{(x-1)(x-3)} = \frac{(A+B)x-3A-B}{(x-1)(x-3)}$$

By equality of left and right side denominators and numerators it follows that

$$
\begin{aligned}
& x+1 \equiv (A+B)x - 3A - B \\
& \quad \therefore \\
& \underline{\text{term } x^1} \ : \ 1 = A + B \ \ \therefore \ \ A = 1 - B \\
& \underline{\text{term } x^0} \ : \ 1 = -3A - B \ \ \therefore \ \ 1 = -3(1-B) - B \\
& \quad \therefore \\
& \underline{B = 2} \ \ \therefore \ \ \underline{A = -1}
\end{aligned}
$$

Finally,

$$
\frac{P(x)}{Q(x)} = \frac{x^3 + x^2 - 16x + 16}{x^2 - 4x + 3} = x + 5 - \frac{1}{x-1} + \frac{2}{x-3}
$$

2.159. Given

$$
\frac{P(x)}{Q(x)} = \frac{x-1}{x^2+x} = \frac{x-1}{x(x+1)}
$$

denominator $Q(x)$ has two unique real zeros at $x = 0$ and $x = -1$, as

$$
\begin{aligned}
\frac{x-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} &= \frac{A(x+1) + Bx}{x(x+1)} \\
\therefore \ \ (A+B)\,x + A &= 1 \\
\therefore \ \ A &= -1 \\
\therefore \ \ A + B = 1 \ \ \therefore \ \ B = 1 - A = \ \therefore \ \ B &= 2
\end{aligned}
$$

Therefore,

$$
\frac{x-1}{x^2+x} = -\frac{1}{x} + \frac{2}{x+1}
$$

2.160. Given

$$
\frac{P(x)}{Q(x)} = \frac{x}{(x+1)(x-4)}
$$

denominator $Q(x)$ has two unique real zeros at $x = -1$ and $x = 4$, as

$$
\begin{aligned}
\frac{x}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4} &= \frac{A(x-4) + B(x+1)}{(x+1)(x-4)} \\
\therefore \ \ (A+B)\,x + (B - 4A) &= x \\
\therefore \ \ A + B = 1 \ \ \therefore \ \ A &= 1 - B \\
\therefore \ \ B - 4A = 0 \ \ \therefore \ \ B - 4(1-B) = 0 \ \ \therefore \ \ B &= \frac{4}{5} \ \ \therefore \ \ A = \frac{1}{5}
\end{aligned}
$$

Therefore,

$$\frac{x}{(x+1)(x-4)} = \frac{1}{5}\frac{1}{x+1} + \frac{4}{5}\frac{1}{x-4}$$

2.161. Given

$$\frac{P(x)}{Q(x)} = \frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$$

denominator $Q(x)$ has two unique real zeros at $x = 1$ and $x = -1$, as

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

$$\therefore \ (A+B)\,x + (A-B) = 1$$

$$\therefore \ A + B = 0 \ \ \therefore \ A = -B$$

$$\therefore \ A - B = 1 \ \ \therefore \ 2A = \ \therefore \ A = \frac{1}{2} \ \ \therefore \ B = -\frac{1}{2}$$

Therefore,

$$\frac{1}{x^2-1} = \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}$$

2.162. Given

$$\frac{P(x)}{Q(x)} = \frac{x+2}{x^3-2x^2} = \frac{x+2}{x^2(x-2)}$$

denominator $Q(x)$ has double zero at $x = 0$ and unique real zero at $x = 2$, as

$$\frac{x+2}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2)+B(x-2)+Cx^2}{x^2(x-2)}$$

$$= \frac{(A+C)x^2 + (B-2A)x - 2B}{x^2(x-2)}$$

$$\therefore \ (A+C) = 0 \ \ \therefore \ A = -C$$

$$\therefore \ B - 2A = 1 \ \ \therefore \ A = \frac{B-1}{2}$$

$$\therefore \ -2B = 2 \ \ \therefore \ B = -1 \ \ \therefore \ A = -1 \ \ \therefore \ C = 1$$

Therefore,

$$\frac{x+2}{x^3-2x^2} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-2}$$

2.163. Given

$$\frac{P(x)}{Q(x)} = \frac{2x^2}{x^4-1} = \frac{2x^2}{(x+1)(x-1)(x^2+1)}$$

where, (x^4-1) term may be factorized by the factor theorem, or as

$$\begin{aligned}(x^4-1) &= \left((x^2)^2-1\right) = \left\{a^2-b^2=(a+b)(a-b)\right\}\\ &= (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1)\end{aligned}$$

Thus, denominator $Q(x)$ has unique real zero at $x=-1$, unique real zero at $x=1$, and two complex zeros, as

$$\begin{aligned}\frac{2x^2}{x^4-1} &= \frac{2x^2}{(x+1)(x-1)(x^2+1)} = \frac{A}{x+1}+\frac{B}{x-1}+\frac{Cx+D}{x^2+1}\\ &= \frac{A\,(x-1)\left(x^2+1\right)+B\,(x+1)\left(x^2+1\right)+(Cx+D)\,(x+1)(x-1)}{(x+1)(x-1)(x^2+1)}\\ &= \frac{(A+B+C)x^3+(-A+B+D)x^2+(A+B-C)x-A+B-D}{(x+1)(x-1)(x^2+1)}\end{aligned}$$

$\therefore$

$$\left.\begin{aligned}A+B+C &= 0\\ -A+B+D &= 2\\ A+B-C &= 0\\ -A+B-D &= 0\end{aligned}\right\} = \left\{\begin{aligned}&(1)-(3)\Rightarrow 2C=0 && \therefore\ C=0\\ &(2)-(4)\Rightarrow 2D=2 && \therefore\ D=1\\ &\therefore\ A+B=0\Rightarrow A=-B\\ &(3)+(4)\Rightarrow 2B-1=0 && \therefore\ B=1/2\end{aligned}\right.$$

which leads to $A=-1/2$. Therefore,

$$\frac{2x^2}{x^2(x-2)} = -\frac{1}{2}\frac{1}{x+1}+\frac{1}{2}\frac{1}{x-1}+\frac{1}{x^2+1}$$

2.164. Given

$$\frac{P(x)}{Q(x)} = \frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} \quad \{\text{see P.2.132}\}$$

denominator $Q(x)$ has unique real zero at $x=1$ and two complex zeros, as

$$\begin{aligned}\frac{1}{x^3-1} &= \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+x+1} = \frac{A(x^2+x+1)+(Bx+C)(x-1)}{(x-1)(x^2+x+1)}\\ &= \frac{(A+B)x^2+(A-B+C)x+A-C}{(x-1)(x^2+x+1)}\end{aligned}$$

$\therefore\ A+B=0\ \ \therefore\ A=-B$

$\therefore\ A-B+C=0\ \ \therefore\ C=2B=-2A$

$$\therefore\ A - C = 1 \Rightarrow 3A = 1\ \ \therefore\ A = \frac{1}{3}\ \ \therefore\ C = -\frac{2}{3}\ \ \therefore\ B = -\frac{1}{3}$$

Therefore,

$$\frac{1}{x^3-1} = \frac{1}{3}\frac{1}{x-1} - \frac{1}{3}\frac{x+2}{x^2+x+1}$$

2.165. Given rational function is transformed into its partial fraction form as,

$$\begin{aligned} x^4+1 &= x^4 + 2x^2 + 1 - 2x^2 = \left(x^2+1\right)^2 - \left(\sqrt{2}x\right)^2 = \left\{a^2 - b^2 = (a-b)(a+b)\right\} \\ &= \left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right) \end{aligned}$$

So that partial fraction is,

$$\begin{aligned} \frac{4}{x^4+1} &= \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} \\ &= \frac{(Ax+B)\left(x^2-\sqrt{2}x+1\right) + (Cx+D)\left(x^2+\sqrt{2}x+1\right)}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)} \\ &= \frac{x^3(A+C) + x^2(\sqrt{2}C + D + B - \sqrt{2}A) + x(A - \sqrt{2}B + C + \sqrt{2}D) + B + D}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)} \end{aligned}$$

which is equivalent to the following system

$$\begin{aligned} A + C &= 0\ \ \therefore\ C = -A \\ -\sqrt{2}A + B + \sqrt{2}C + D &= 0 \\ A - \sqrt{2}B + C + \sqrt{2}D &= 0 \\ B + D &= 4\ \ \therefore\ D = 4 - B \end{aligned}$$

$$\begin{aligned} -\sqrt{2}A + \cancel{B} - \sqrt{2}A + 4 - \cancel{B} = 0\ \ &\therefore\ A = \sqrt{2}\ \ \therefore\ C = -\sqrt{2} \\ \cancel{A} - \sqrt{2}B - \cancel{A} + 4\sqrt{2} - \sqrt{2}B = 0\ \ &\therefore\ B = 2\ \ \therefore\ D = 2 \end{aligned}$$

2.166. Given

$$\frac{P(x)}{Q(x)} = \frac{1}{(x+1)(x+2)}$$

denominator $Q(x)$ has two unique real zeros at $x = -1$ and $x = -2$, as

$$\begin{aligned} \frac{1}{(x+1)(x+2)} &= \frac{A}{x+1} + \frac{B}{x+2} \\ &= \frac{A(x+2) + B(x+1)}{(x+1)(x+2)} \end{aligned}$$

$$\therefore \quad (A+B)\,x + (2A+B) = 1$$

$$\therefore \quad A + B = 0 \quad \therefore \quad A = -B$$

$$\therefore \quad 2A + B = 1 \quad \therefore \quad -B = \quad \therefore \quad B = -1 \quad \therefore \quad A = 1$$

Therefore,

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

Linear Equations and Inequalities

3

Problems

3.1 Linear Equations

Solve and comment on the solutions of equations in P.3.1 to P.3.6.

3.1. $1-\dfrac{x-1}{3}=\dfrac{3-x}{3}$ **3.2.** $3x+14=5x-2\,(x-7)$ **3.3.** $\dfrac{1+3x}{4}-\dfrac{6x+3}{12}=\dfrac{x}{4}$

3.4. $\dfrac{x}{x-2}-\dfrac{2x+3}{x+2}=\dfrac{x^2}{4-x^2}$ **3.5.** $\dfrac{1}{x-1}=\dfrac{x^2}{4}$ **3.6.** $\dfrac{x}{x-2}-2\,\dfrac{2x+3}{2x+1}=-\dfrac{7x}{3}$

Solve "telescopic" forms of equations in P.3.7 to P.3.9.

3.7. $1-\dfrac{1}{1-\frac{1}{1-x}}=\dfrac{1}{2}$ **3.8.** $\dfrac{1}{2+\frac{1}{2+\frac{1}{2-x}}}=\dfrac{5}{12}$ **3.9.** $\dfrac{1}{2+\frac{1}{3+\frac{1}{4+\frac{3}{x+1}}}}=\dfrac{16}{37}$

3.2 System of Linear Equations

Solve systems of equations in P.3.10 to P.3.13.

3.10. $\begin{cases} x+2y &= 2x-5 \\ x-y &= 3 \end{cases}$ **3.11.** $\begin{cases} xy &= 10 \\ x+y &= 2 \end{cases}$

3.12. $\begin{cases} \dfrac{14}{x}+\dfrac{24}{y} &= 10 \\ \dfrac{7}{x}-\dfrac{18}{y} &= -5 \end{cases}$ **3.13.** $\begin{cases} \dfrac{4}{x+y-1}+\dfrac{1}{x-y+1} &= 1 \\ \dfrac{16}{x+y-1}-\dfrac{2}{(x-y+1)} &= 1 \end{cases}$

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023
R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41200-4_3

3.3 Linear Inequalities

Solve inequalities in P.3.14 to P.3.22.

3.14. $(x+3)^2 \leq 10x+6$

3.15. $(x+5)(x-5) > 0$

3.16. $\dfrac{1}{x} + \dfrac{1}{x+1} < \dfrac{x^2-2}{x^2+x}$

3.17. $-3x^2+30x-75 > 0$

3.18. $x^4 - 2x^2 + 1 < 0$

3.19. $-2x^2+4x-2 > 0$

3.20. $\dfrac{2x^2-5x-3}{-2x+1} \leq 0$

3.21. $\left|\dfrac{4x}{2x+4}\right| < 3$

3.22. $\dfrac{(x+2)\,|x-2|}{x^2+2} > 1$

3.4 System of Linear Inequalities

Solve systems of inequalities in P.3.23 to P.3.28.

3.23. $\begin{cases} y > 2x \\ y < -x-3 \end{cases}$

3.24. $\begin{cases} -2(x+1) \geq 4 \\ -2x \geq 6 \\ x \leq -3 \end{cases}$

3.25. $\begin{cases} y \geq 2x\,; & y \leq 2x \\ x \leq 4\,; & x \geq 4 \end{cases}$

3.26. $\begin{cases} -4x+6y > 6 \\ 2x-3y > 3 \end{cases}$

3.27. $\begin{cases} -2x-y < -1 \\ 4x+2y \leq 6 \end{cases}$

3.28. $\begin{cases} x+y < 6 \\ 5x+y \geq 4 \\ x+4y > -3 \\ x \leq 5 \\ y < 4 \end{cases}$

Answers

3.1 Linear Equations

Solution to linear equation (i.e., with a single variable) is a single number. If the variable is canceled however, then there may be either infinity of solutions (i.e., an identity) or no solution (i.e., impossible equation). Sometimes, higher-order polynomial terms may cancel, leaving only the linear part of the equation.

3.1. After a simple calculation,

$$1-\frac{x-1}{3}=\frac{3-x}{3} \;\;\therefore\;\; \frac{3-x+1}{3}=\frac{3-x}{3} \;\;\therefore\;\; 4-\cancel{x}=3-\cancel{x} \;\;\therefore\;\; 4\neq 3$$

Conclusion: this equation is not possible; there are no valid solution regardless of x.

3.2. After a simple calculation,

$$3x+14=5x-2\,(x-7) \;\;\therefore\;\; \cancel{3x}+\cancel{14}-\cancel{3x}-\cancel{14}=0 \;\;\therefore\;\; 0=0$$

Conclusion: this equation is true for any x; thus, there are infinitely many solutions.

3.3. After a simple calculation,

$$\frac{1+3x}{4}-\frac{6x+3}{12}=\frac{x}{4} \;\;\therefore\;\; \frac{\cancel{3}+9x-6x-\cancel{3}}{\cancel{12}}=\frac{3x}{\cancel{12}} \;\;\therefore\;\; \cancel{3}x=\cancel{3}x$$
$$\therefore\;\; x=x \;\;\therefore\;\; \cancel{x}-\cancel{x}=0 \;\;\therefore\;\; 0=0$$

Conclusion: this equation is an identity that is true for any x; thus, there are infinitely many solutions.

3.4. In this example, the equation is not defined for $x=\pm 2$, because for each of the three rational terms, either $x=+2$ or $x=-2$ results in polynomial division by zero. Then

$$\frac{x}{x-2}-\frac{2x+3}{x+2}=\frac{x^2}{4-x^2} \;\;\therefore\;\; \frac{x(x+2)-(2x+3)(x-2)}{(x-2)(x+2)}=\frac{x^2}{4-x^2}$$
$$\therefore\;\; \frac{-x^2+3x+6}{x^2-4}=\frac{x^2}{4-x^2}$$
$$\frac{x^2-3x-6}{4-x^2}=\frac{x^2}{4-x^2}$$

After factorizations and simplifications, denominator on the left side equals to denominator on the right side; thus, it must be true that the two numerators are also equal, i.e.,

$$x^2-3x-6=x^2$$
$$\cancel{x^2}-3x-6-\cancel{x^2}=0$$
$$-3x-6=0 \;\;\therefore\;\; x=-2$$

However, value of the solution $x = -2$ is already excluded. Thus, the conclusion is that this example does not have valid solutions.

3.5. Note that given equation is not defined for $x = 1$ due to division by zero on the left side,

$$\frac{1}{x-1} = \frac{x^2}{4} \quad \therefore \quad 4 = x^2(x-1) \quad \therefore \quad x^3 - x^2 - 4 = 0$$

A third-order polynomial must have at least one zero, here by inspection at $x = 2$; thus, by long polynomial division

$$\begin{aligned}
(x^3 - x^2 - 4) \div (x-2) &= x^2 + x + 2 \\
-\underline{(x^3 - 2x^2)} & \\
x^2 - 4 & \\
-\underline{(x^2 - 2x)} & \\
2x - 4 & \\
-\underline{(2x - 4)} & \\
0 &
\end{aligned}$$

where $x^2 + x + 2$ does not have real zeros (check its discriminant). Thus, $x = 2$ is the only real solution to the given equation; the other two are complex.

3.6. In this example, given equation is not defined for $x = +2$ or $x = -1/2$ because it results in polynomial division by zero. Then

$$\begin{aligned}
\frac{x}{x-2} - 2\,\frac{2x+3}{2x+1} = -\frac{7x}{3} \quad \therefore \quad \frac{x(2x+1) - 2(2x+3)(x-2)}{(x-2)(2x+1)} &= -\frac{7x}{3} \\
\therefore \quad \frac{-2x^2 + 3x + 12}{(x-2)(2x+1)} &= -\frac{7x}{3} \\
3(-2x^2 + 3x + 12) &= -7x(x-2)(2x+1) \\
-6x^2 + 9x + 36 &= -14x^3 + 21x^2 + 14x \\
14x^3 - 27x^2 - 5x + 36 &= 0
\end{aligned}$$

Third-order polynomial must have at least one real zero, here by inspection at $x = -1$. Long polynomial division gives

$$\begin{aligned}
(14x^3 - 27x^2 - 5x + 36) \div (x+1) &= 14x^2 - 41x + 36 \\
-\underline{(14x^3 + 14x^2)} & \\
-41x^2 - 5x & \\
-\underline{(-41x^2 - 41x)} &
\end{aligned}$$

$$\begin{aligned} & 36x + 36 \\ -\ & \underline{(36x + 36)} \\ & = 0 \end{aligned}$$

where $14x^2 - 41x + 36$ does not have real zeros (check its discriminant). Thus, $x = -1$ is the only real solution to the given equation.

3.7. "Unfold" this form by add fraction and then invert repetition.

$$1 - \frac{1}{1 - \frac{1}{1-x}} = \frac{1}{2} \ \therefore \ \frac{1}{1 - \frac{1}{1-x}} = 1 - \frac{1}{2} = \frac{1}{2} \ \therefore \ \{\text{invert both sides}\}$$

$$\therefore \ 1 - \frac{1}{1-x} = 2 \ \therefore \ 1 - 2 = \frac{1}{1-x}$$

$$\therefore \ 1 - x = -1$$

$$\therefore \ x = 2$$

3.8. "Unfold" this form by invert and then add fractions repetition

$$\frac{1}{2 + \frac{1}{2 + \frac{1}{2-x}}} = \frac{5}{12} \ \therefore \ \{\text{invert both sides}\} \ \therefore \ 2 + \frac{1}{2 + \frac{1}{2-x}} = \frac{12}{5} \ \therefore \ \frac{1}{2 + \frac{1}{2-x}} = \frac{2}{5}$$

$$\therefore \ 2 + \frac{1}{2-x} = \frac{5}{2} \ \therefore \ \frac{1}{2-x} = \frac{1}{2}$$

$$\therefore \ 2 - x = 2$$

$$\therefore \ x = 0$$

3.9. "Unfold" this form by invert and then add fractions repetition

$$\frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \frac{3}{x+1}}}} = \frac{16}{37} \ \therefore$$

$$2 + \frac{1}{3 + \frac{1}{4 + \frac{3}{x+1}}} = \frac{37}{16} \ \therefore$$

$$\frac{1}{3 + \frac{1}{4 + \frac{3}{x+1}}} = \frac{37}{16} - 2$$

$$\therefore \Rightarrow$$

$$\Rightarrow \ \therefore$$

$$\frac{1}{3 + \frac{1}{4 + \frac{3}{x+1}}} = \frac{5}{16} \ \therefore$$

$$3 + \frac{1}{4 + \frac{3}{x+1}} = \frac{16}{5} \ \therefore$$

$$\frac{1}{4 + \frac{3}{x+1}} = \frac{16}{5} - 3 = \frac{1}{5} \ \therefore$$

$$4 + \frac{3}{x+1} = 5 \ \therefore$$

$$\frac{3}{x+1} = 1$$

Therefore, $x + 1 = 3 \ \therefore \ x = 2$.

3.2 System of Linear Equations

Geometrical interpretation of solution to a system of two linear equations with two unknowns is that it represents spatial coordinates (x, y) of the intersection point of these two lines in the plane. That is to say, the only point that is shared by two lines. If two lines are not parallel, there must be one intersection point. However, if two lines are parallel, either all points are the solution (i.e., the two lines are superimposed) or none of the points are common (i.e., the two lines are parallel and separated by a distance). In the following examples, only the real solutions (i.e., only those that can be shown in xy plane) are illustrated.

3.10. This simple system is easily resolved by the substitution method as

$$\begin{cases} x + 2y &= 2x - 5 \\ x - y &= 3 \quad \Rightarrow x = 3 + y \end{cases}$$

$$\therefore \quad x + 2y = 2x - 5 \quad \therefore \quad 3 + y + 2y = 2(3 + y) - 5 \quad \therefore \quad 3y + 3 = 2y + 1$$

$$\therefore \quad y = -2 \quad \text{and} \quad x = 1$$

Thus, the solution of this system is $(x, y) = (1, -2)$ point. With simple transformations, this system may be written explicitly as two linear functions and then show their intersect point (see Fig. 3.1).

$$x + 2y = 2x - 5 \Rightarrow y = \frac{x - 5}{2}$$

$$x - y = 3 \Rightarrow y = x - 3$$

After equalizing the two equations, it follows as same as in the previous method.

$$\frac{x - 5}{2} = x - 3 \Rightarrow x - 5 = 2x - 6 \quad \therefore \quad x = 1 \Rightarrow y = -2$$

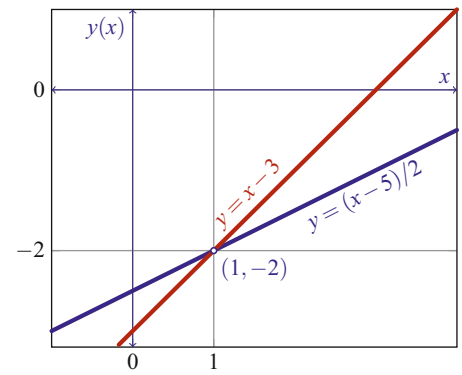
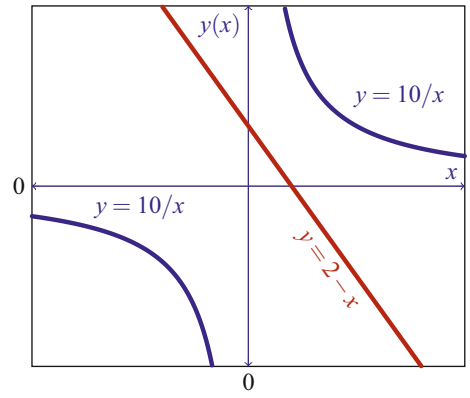


Fig. 3.1 Example P.3.10

Fig. 3.2 Example P.3.11



3.11. By the substitution method,

$$\begin{cases} xy & = 10 \quad \Rightarrow y = \dfrac{10}{x} \\ x + y & = 2 \quad \Rightarrow y = 2 - x \end{cases}$$

$$\therefore \; x + \frac{10}{x} = 2 \;\; \therefore \;\; x^2 - 2x + 10 = 0$$

Note that form of this system (where there is product of xy) is a quadratic equation in disguise. In this case, the resulting quadratic equation does not have real solutions due to its discriminant being $\Delta = (-2)^2 - 4 \cdot 1 \cdot 10 = -36 < 0$. Thus, there are no real solutions to this system. In order to illustrate that there are no intersect points, two equations are shown in graph that clearly show no intersect point (see Fig. 3.2).

3.12. Given system in the form of $f(1/x, 1/y)$ that may appear to be nonlinear, the inverse functions of two original variables may be replaced by new variables, so that this nonlinear system relative to (x, y) is transformed into linear system relative to (t, k) as

$$\begin{cases} \dfrac{14}{x} + \dfrac{24}{y} & = 10 \\ \dfrac{7}{x} - \dfrac{18}{y} & = -5 \end{cases} \;\; \therefore \;\; \left\{ t = \frac{1}{x}, \;\; k = \frac{1}{y} \right\} \;\; \therefore \;\; \begin{cases} 14t + 24k & = 10 \\ 7t - 18k & = -5 \end{cases} \; \{\text{expand by } (\times 2)\}$$

$$\Rightarrow \begin{cases} 14t + 24k & = 10 \\ 14t - 36k & = -10 \end{cases} \;\; \therefore \;\; \{Eq.(1) - -(2)\} \;\; \therefore \;\; 60k = 20 \;\; \therefore \;\; k = \frac{1}{3} \Rightarrow$$

$$\underline{y = \frac{1}{k} = 3} \;\; \therefore \;\; 14t + 24\left(\frac{1}{3}\right) = 10 \;\; \therefore \;\; 14t + 8 = 10 \;\; \therefore \;\; t = \frac{1}{7} \;\; \therefore \;\; \underline{x = \frac{1}{t} = 7}$$

Thus, the solution of this system is $(x, y) = (7, 3)$ point.

After simple transformations, the original nonlinear system may be written as two explicit functions of x as,

$$\frac{14}{x} + \frac{24}{y} = 10 \;\; \Rightarrow \;\; y = \frac{12x}{5x - 7}$$

$$\frac{7}{x}-\frac{18}{y}=-5 \;\Rightarrow\; y=\frac{18x}{7+5x}$$

And linear version of the system may be written as

$$7t+12k=5 \;\Rightarrow\; k=\frac{5-7t}{12}$$
$$7t-18k=-5 \;\Rightarrow\; k=\frac{7t+5}{18}$$

Both pairs of functions and their intersection points are shown by graphs, nonlinear system in Fig. 3.3 (left) and its equivalent linear system in Fig. 3.3 (right).

3.13. By the change of variables method, two denominators are replaced as

$$\begin{cases} \dfrac{4}{x+y-1}+\dfrac{1}{x-y+1} &= 1 \\ \dfrac{16}{x+y-1}-\dfrac{2}{x-y+1} &= 1 \end{cases} \quad \therefore \quad \left\{ t=\frac{1}{x+y-1},\;\; k=\frac{1}{x-y+1} \right\}$$

$$\therefore$$

$$\begin{cases} 4t+k &= 1 \;\Rightarrow t=\dfrac{1-k}{4} \\ 16t-2k &= 1 \end{cases}$$

$$\therefore$$

$$16\,\frac{1-k}{4}-2k=1 \;\;\therefore\;\; 4-6k=1 \;\;\therefore\;\; k=\frac{1}{2} \;\;\therefore\;\; \frac{1}{k}=2$$

$$\therefore\;\; 4t+\left(\frac{1}{2}\right)=1 \;\;\therefore\;\; 4t=\frac{1}{2} \;\;\therefore\;\; t=\frac{1}{8} \;\;\therefore\;\; \frac{1}{t}=8$$

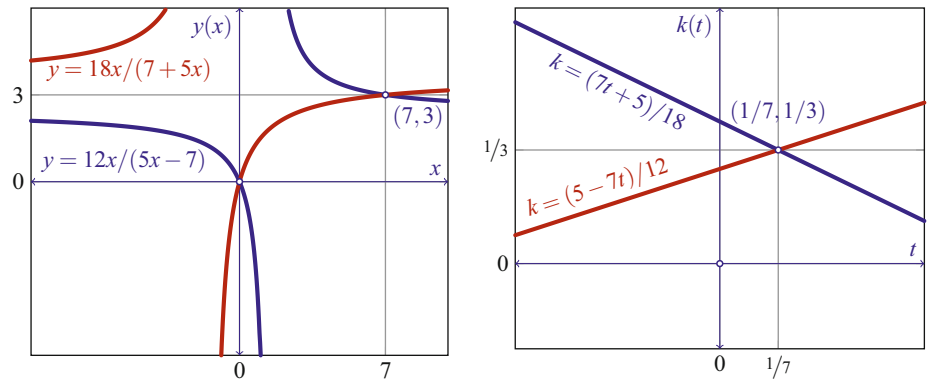


Fig. 3.3 Example P.3.12

After returning back the original variables, there is the following system of equations:

$$\begin{cases} \frac{1}{t} = x + y - 1 & = 8 \\ \frac{1}{k} = x - y + 1 & = 2 \end{cases} \therefore \begin{cases} x + y & = 9 \\ x - y & = 1 \end{cases} \Rightarrow x = y + 1 \quad \therefore \quad \underline{x = 5, \quad y = 4}$$

Therefore, solution to this system is $(x, y) = (5, 4)$ point.

3.3 Linear Inequalities

Solution to inequalities may be one or more *intervals*, one or more points, or no solution at all.

3.14. Given inequality may be resolved similarly to equations by balancing the left and right sides. This time however, it is very important to balance correctly $\leq$ and $\geq$ sides (which was not an issue in the case of simple equations), as

$$\begin{aligned} (x+3)^2 &\leq 10x + 6 \\ x^2 + 6x + 9 &\leq 10x + 6 \\ x^2 - 4x + 3 &\leq 0 \\ x^2 - 3x - x + 3 &\leq 0 \\ x(x-3) - (x-3) &\leq 0 \\ (x-1)(x-3) &\leq 0 \end{aligned}$$

Reminder: Product of two numbers is negative only if the two numbers have opposite signs, i.e., if $AB < 0$, then either $A < 0$ and $B > 0$ or $A > 0$ and $B < 0$.

In this case,

$$(x-1)(x-3) \leq 0 \quad \Rightarrow \quad \begin{cases} x - 1 & \leq 0 \quad \therefore \quad x \leq 1 \\ x - 3 & \geq 0 \quad \therefore \quad x \geq 3 \\ & \text{or,} \\ x - 1 & \geq 0 \quad \therefore \quad x \geq 1 \\ x - 3 & \leq 0 \quad \therefore \quad x \leq 3 \end{cases}$$

First of the two possibilities results in no solutions, i.e., if $x \leq 1$ and $x \geq 3$, then there is no x that satisfies both conditions at the same time. Second of the two possibilities, i.e., if $x \geq 1$ and $x \leq 3$, then the solution is $x \in (1, 3)$ interval, where both $x = 1$ and $x = 3$ points are included in the solution set. Solutions of inequalities may be shown in graph form as shaded intervals, where solid boundary lines and crossed circles indicate that the associated boundary point *is* included, while

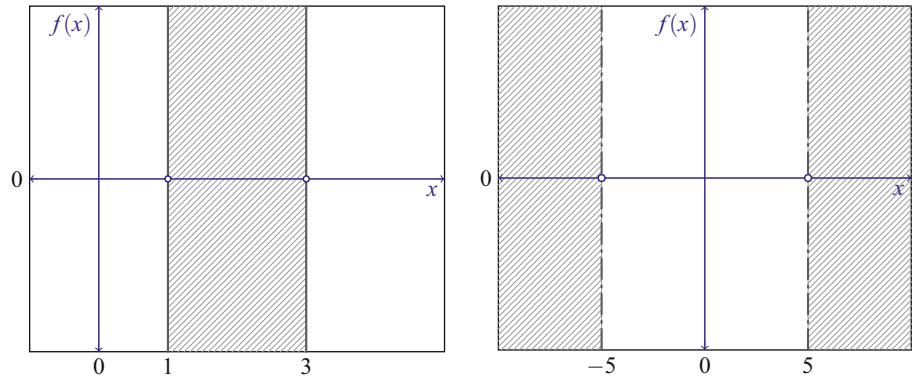


Fig. 3.4 Examples P.3.14, P.3.15

dashed boundary lines and non-crossed circles indicate that the boundary point is *not* included in the solution set. In this case, both boundary points are included (due to non-strict inequality "≤") (see Fig. 3.4 (left)).

3.15. Given factorized form,

$$(x+5)(x-5) > 0$$

In this case,

$$(x+5)(x-5) > 0 \;\Rightarrow\; \begin{cases} x+5 > 0 \;\therefore\; x > -5 \\ x-5 > 0 \;\therefore\; x > 5 \\ \quad \text{or,} \\ x+5 < 0 \;\therefore\; x < -5 \\ x-5 < 0 \;\therefore\; x < 5 \end{cases}$$

Therefore, first two conditions $x > -5$ and $x > 5$ are satisfied at the same time if $x > 5$. Similarly, conditions $x < -5$ and $x < 5$ are satisfied at the same time if $x < -5$.

In summary,

$$(x+5)(x-5) > 0 \;\Rightarrow\; \underline{x < -5 \quad \text{or} \quad x > 5}$$

Note that in this case, both boundary points ($x = -5$ and $x = 5$) are *not* included (due to strict inequalities "<>") (see Fig. 3.4 (right)).

3.16. Given inequality of two rational functions, it is noted that $x \neq 0$ and $x \neq -1$ are to be excluded due to division by zero, as

$$\frac{1}{x} + \frac{1}{x+1} < \frac{x^2-2}{x^2+x} = \frac{x^2-2}{x(x+1)}$$

With that in mind, this inequality may be converted to the form of inequality relative to zero as

$$\begin{aligned}
\frac{1}{x}+\frac{1}{x+1} &< \frac{x^2-2}{x(x+1)} \\
\frac{x+1+x}{x(x+1)} &< \frac{x^2-2}{x(x+1)} \\
0 &< \frac{x^2-2}{x(x+1)}-\frac{2x+1}{x(x+1)} \\
0 &< \frac{x^2-2-2x-1}{x(x+1)} \\
0 &< \frac{x^2-2x-3}{x(x+1)} \\
0 &< \frac{x^2+x-3x-3}{x(x+1)} \\
0 &< \frac{x(x+1)-3(x+1)}{x(x+1)} \\
0 &< \frac{\cancel{(x+1)}(x-3)}{x\cancel{(x+1)}} \quad \{(x\neq -1)\} \\
&\therefore \\
\frac{x-3}{x} &> 0
\end{aligned}$$

Note how left- and right-side expressions must keep correct "<>" relationship when changing the sides. (Still, $x \neq 0$ and $x \neq -1$.)

The last rational function is positive only when both numerator and denominator have the same sign.

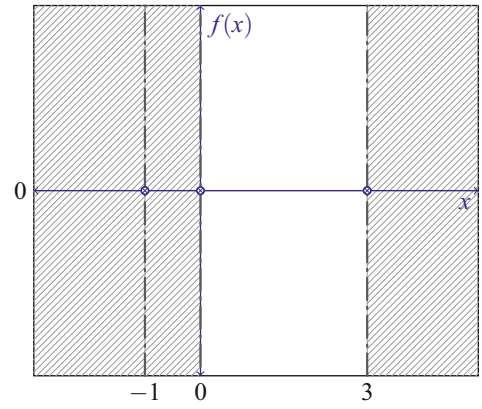
$$\frac{x-3}{x} > 0 \quad \Rightarrow \quad \begin{cases} x-3 & > 0 \quad \therefore \quad x > 3 \\ x & > 0 \quad \therefore \quad x > 0 \\ & \text{or,} \\ x-3 & < 0 \quad \therefore \quad x < 3 \\ x & < 0 \quad \therefore \quad x < 0 \end{cases}$$

Therefore, first two conditions $x > 3$ and $x > 0$ are satisfied at the same time if $x > 3$. Similarly, conditions $x < 3$ and $x < 0$ are satisfied at the same time if $x < 0$.

In summary,

$$\frac{x-3}{x} > 0 \quad \Rightarrow \quad \underline{x < 0 \quad \text{or} \quad x > 3} \quad \{(x \neq -1)\}$$

Fig. 3.5 Example P.3.16



Note that in this case, boundary point ($x = 0$) is *not* included (due to strict inequality "$<$" and division by zero), boundary point ($x = 3$) is *not* included due to strict inequality "$>$," and ($x = -1$) is *not* included due to division by zero (see Fig. 3.5).

3.17. Given,

$$-3x^2 + 30x - 75 > 0 \quad \therefore \quad -3(x^2 - 10x + 25) > 0$$
$$-3(x^2 - 5x - 5x + 25) > 0$$
$$-3[x(x-5) - 5(x-5)] > 0$$
$$\therefore$$
$$\underbrace{-3}_{<0} \quad \underbrace{(x-5)^2}_{>0} > 0 \quad \therefore \quad \text{not possible}$$

because product of negative and positive terms is always negative.

3.18. Given,

$$x^4 - 2x^2 + 1 < 0$$
$$(x^2)^2 - 2x^2 + 1 < 0$$
$$(x^2 - 1)^2 < 0$$
$$\underbrace{(x+1)^2}_{\geq 0} \quad \underbrace{(x-1)^2}_{\geq 0} < 0 \qquad \therefore \quad \text{not possible}$$

3.19. Given,

$$-2x^2 + 4x - 2 > 0$$
$$-2x^2 + 4x - 2 > 0$$

$$-2(x^2 - 2x + 1) > 0$$

$$\underbrace{-2}_{<0}\ \underbrace{(x-1)^2}_{>0} > 0 \qquad \therefore \quad \text{not possible}$$

because product of one negative and one positive terms is always negative.

3.20. Given

$$\frac{2x^2 - 5x - 3}{-2x + 1} \leq 0$$

$$\frac{2x^2 - 6x + x - 3}{-2x + 1} \leq 0$$

$$\frac{2x(x-3) + (x-3)}{-2x + 1} \leq 0$$

$$\frac{(x-3)(2x+1)}{-2x + 1} \leq 0$$

Note that division by zero occurs when $x = 1/2$; thus, it must be excluded. This inequality is possible only if numerator and denominator have opposite signs. That may be summarized systematically by examining sign of each factorized term and the total sign of the rational term as

x		$-1/2$		$1/2$		3	
$x - 3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$2x + 1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$-2x + 1$	$+$	$+$	$+$	0	$-$	$-$	$-$
Total sign	$+$	0	$-$	n.d.	$+$	0	$-$

In conclusion

$$\frac{(2x+1)(x-3)}{-2x+1} \leq 0 \quad \therefore \quad -\frac{1}{2} \leq x < \frac{1}{2} \quad \text{and} \quad x \geq 3$$

Note that in this case, boundary point ($x = -1/2$) is included (due to non-strict inequality "$\leq$"), boundary point ($x = 1/2$) is *not* included due to division by zero, and ($x = 3$) is included due to non-strict inequality "$\geq$" (see Fig. 3.6).

3.21. Absolute value terms are converted into two inequalities

$$\left|\frac{4x}{2x+4}\right| < 3 \quad \therefore \quad -3 < \frac{4x}{2x+4} < 3 \quad \Rightarrow$$

$$\frac{4x}{2x+4} < 3 \quad \therefore \quad 4x < 6x + 12 \quad \therefore \quad -2x < 12 \quad \therefore \quad \underline{x < -6}$$

$$\frac{4x}{2x+4} > -3 \quad \therefore \quad 4x > -6x - 12 \quad \therefore \quad 10x > -12 \quad \therefore \quad \underline{x > -\frac{6}{5}}$$

Fig. 3.6 Example P.3.20

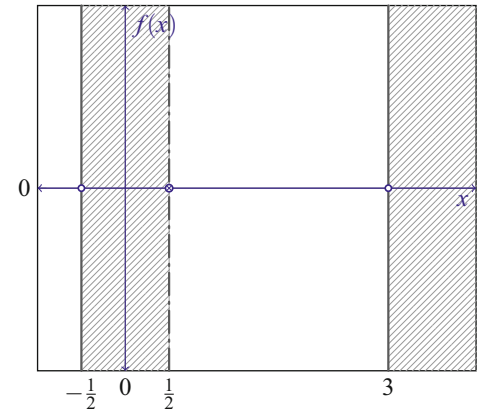
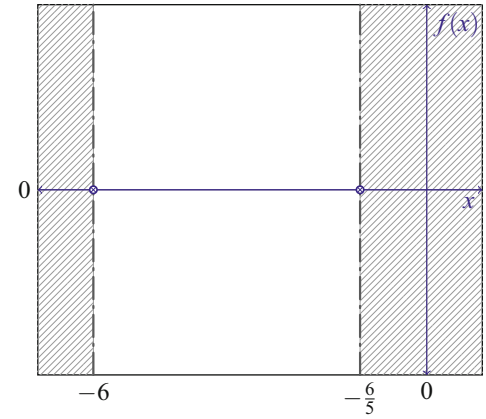


Fig. 3.7 Example P.3.21



Note that in this case, both boundary points, $x = -6$ and $x = -6/5$, are *not* included (due to strict inequality "<>") (see Fig. 3.7).

3.22. Absolute value terms are converted into two inequalities,

$$\frac{(x+2)\,|x-2|}{x^2+2} > 1 \quad \left\{ \begin{array}{llll} 1) & x-2 & \geq 0 \;\therefore\; x \geq 2 & \therefore \;\; |x-2| = x-2 \\ 2) & x-2 & < 0 \;\therefore\; x < 2 & \therefore \;\; |x-2| = -(x-2) \end{array} \right\}$$

$(x \geq 2)$:

Method 1:

$$\frac{(x+2)(x-2)}{x^2+2} > 1 \;\; \therefore \;\; \frac{x^2-4}{x^2+2} > 1 \;\; \therefore \;\; \frac{\cancel{x^2}\left(1-\frac{4}{x^2}\right)}{\cancel{x^2}\left(1+\frac{2}{x^2}\right)} > 1 \;\; \times$$

Method 2:

$$\frac{x^2-4}{x^2+2} > 1$$
$$x^2 - 4 > x^2 + 2$$
$$\cancel{x^2} - 4 - \cancel{x^2} - 2 > 0$$
$$-6 > 0 \quad \times$$

In Method 1, numerator is inferior to "1," and denominator is greater than "1"; thus their ratio is always less than "1," not greater. In Method 2, it is obvious that "−6" is negative, not positive. In conclusion, for $(x \geq 2)$, there is no solution to this inequality.

$(x < 2)$:

$$\frac{-(x+2)(x-2)}{x^2+2} > 1$$
$$\frac{4-x^2}{x^2+2} > 1$$
$$4 - x^2 > x^2 + 2$$
$$4 - 2 > 2x^2$$
$$\therefore$$
$$1 > x^2 \Rightarrow \sqrt{x^2} < \sqrt{1} \quad \therefore \quad |x| < 1$$
$$\therefore$$
$$-1 < x < +1$$

Note that both boundary points ($x = \pm 1$) are *not* included (due to strict inequality "<") (see Fig. 3.8).

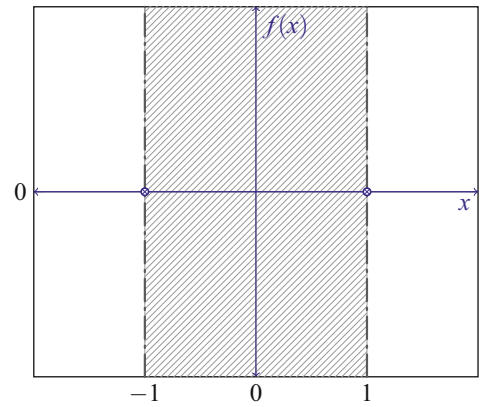


Fig. 3.8 Example P.3.22

3.4 System of Linear Inequalities

3.23. Given system of inequalities,

$$y > 2x$$
$$y < -x - 3$$

the solution sets for each inequality may be shown graphically in Fig. 3.9.

Then, the superimposed region represents the solution set of this system of inequalities (see Fig. 3.10 (left)). Note that the boundary lines are not included in the solution set (due to strict non-equalities "<>"). In addition, coordinates of the intersection point $\mathcal{A} = (-1, -2)$ are found as

$$2x = -x - 3 \;\;\therefore\;\; 3x = -3 \;\;\therefore\;\; x = -1 \;\;\therefore\;\; y = 2x = -2 \;\; (\text{or, } y = -x - 3 = -2)$$

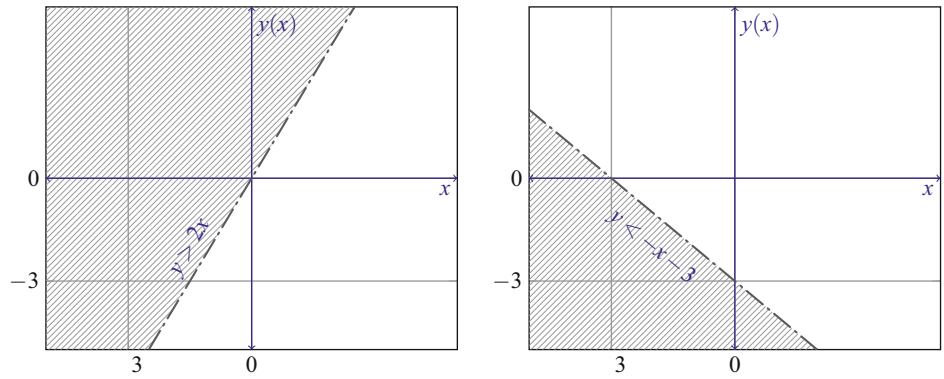


Fig. 3.9 Example P.3.23

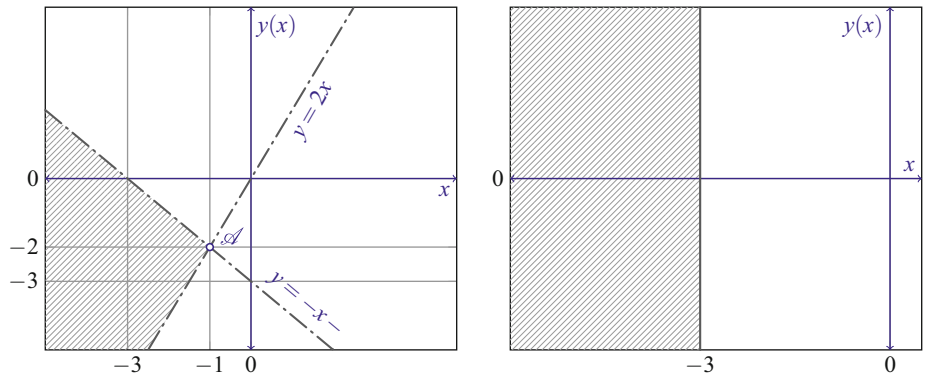


Fig. 3.10 Examples P.3.23, P.3.24

3.24. Given system of inequalities

$$-2(x+1) \geq 4 \;\; \therefore \;\; -x-1 \geq 2 \;\; \therefore \;\; -3 \geq x \;\; \therefore \;\; \underline{x \leq -3}$$
$$-2x \geq 6 \;\; \therefore \;\; -x \geq 3 \;\; \therefore \;\; -3 \geq x \;\; \therefore \;\; \underline{x \leq -3}$$
$$x \leq -3 \;\; \therefore \;\; \underline{x \leq -3}$$

all three inequalities reduce to the same ($x \leq -3$) interval (see Fig. 3.9 (right)).

3.25. Given system of inequalities,

$$y \geq 2x\,; \quad y \leq 2x$$
$$x \leq 4\,; \quad x \geq 4$$

The first two inequalities, $y \geq 2x$ and $y \leq 2x$, cover space both above and below $y = 2x$ line, thus the *whole space*. Note that $y = 2x$ line is included in the solution set. The second two inequalities, however, add constrains that $x \leq 4$ and $x \geq 4$ at the same time. Therefore, there is only one point $x = 4$ that satisfies all four inequalities, and consequently $y = 2x = 8$. The solution set is a single point $(4, 8)$.

3.26. Given system of nonlinearities

$$-4x + 6y > 6 \;\; \therefore \;\; y > \frac{6+4x}{6} = \frac{2}{3}x + 1$$
$$2x - 3y > 3 \;\; \therefore \;\; \frac{2x-3}{3} > y \;\; \therefore \;\; y < \frac{2}{3}x - 1$$

is solved graphically as in Fig. 3.11. Obviously, there is no overlapping region; thus, the solution set is empty. Note that the two boundary lines are not included in the solution due to strict "<>" inequalities.

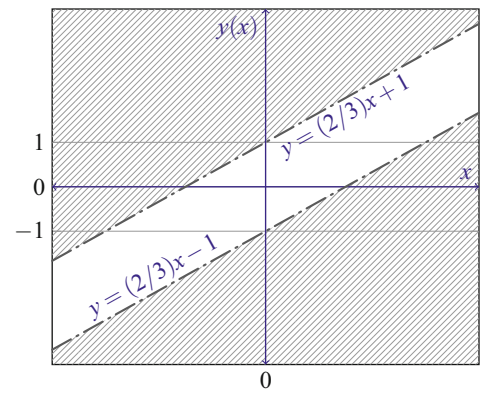


Fig. 3.11 Example P.3.26

3.27. Given system of inequalities,

$$-2x - y < -1 \;\;\therefore\;\; -2x + 1 < y \;\;\therefore\;\; y > -2x + 1$$
$$4x + 2y \le 6 \;\;\therefore\;\; 2y \le 6 - 4x \;\;\therefore\;\; y \le -2x + 3$$

graphical solution is in Fig. 3.12. Note that $y = -2x + 1$ line is not in the solution set due to strict inequality.

3.28. Given system of inequalities,

$$x + y < 6 \;\;\therefore\;\; y < 6 - x$$
$$5x + y \ge 4 \;\;\therefore\;\; y \ge 4 - 5x$$
$$x + 4y > -3 \;\;\therefore\;\; y > -\frac{3}{4} - \frac{1}{4}x$$
$$x \le 5$$
$$y < 4$$

graphical solution is in Fig. 3.12 (right). The solution set is bound by these five linear functions, where the intersection points are:

$\mathcal{A}$: $y = 4$ and $y = -5x + 4 \;\;\therefore\;\; -5x + 4 = 4 \;\;\therefore\;\; -5x = 0 \;\;\therefore\;\; \underline{x = 0} \;\Rightarrow (0, 4)$

$\mathcal{B}$: $y = 4$ and $y = -x + 6 \;\;\therefore\;\; -x + 6 = 4 \;\;\therefore\;\; \underline{x = 2} \;\Rightarrow (2, 4)$

$\mathcal{C}$: $x = 5$ and $y = -x + 6 \;\;\therefore\;\; y = -5 + 6 \;\;\therefore\;\; \underline{y = 1} \;\Rightarrow (5, 1)$

$\mathcal{D}$: $x = 5$ and $y = (-x - 3)/4 \;\;\therefore\;\; y = (-5 - 3)/4 \;\;\therefore\;\; \underline{y = -2} \;\Rightarrow (5, -2)$

$\mathcal{E}$: $y = (-x - 3)/4$ and $y = -5x + 4 \;\;\therefore\;\; -5x + 4 = (-x - 3)/4 \;\;\therefore\;\; -20x + 16 = -x - 3$
$\therefore\;\; \underline{x = 1} \;\;\therefore\;\; y = -5 + 4 \;\;\therefore\;\; \underline{y = -1} \;\Rightarrow (1, -1)$

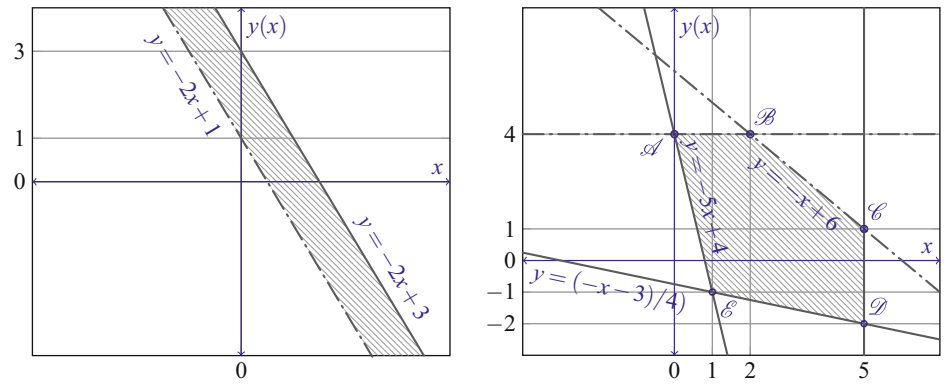


Fig. 3.12 Examples P.3.27, P.3.28

Note that due to non-strict inequalities, the following lines and points are *not* part of the solution set:

$$y = 4$$

$$y = -x + 6$$

$\mathcal{A}$ and $\mathcal{B}$ (they belong to $y = 4$)

$\mathcal{C}$ (it belongs to $y = -x + 6$ line)

Check for updates

Logarithmic and Exponential Functions

4

Problems

4.1 Logarithmic and Exponential Functions

Sketch graphs, and show important points of basic exponential and logarithmic functions in P.4.1 to P.4.12

4.1. $f(x) = 2^x$ **4.2.** $f(x) = e^x$ **4.3.** $f(x) = 10^x$

4.4. $f(x) = \log_2(x)$ **4.5.** $f(x) = \ln(x)$ **4.6.** $f(x) = \log(x)$

4.7. $f(x) = 2^{-x}$ **4.8.** $f(x) = e^{-x}$ **4.9.** $f(x) = 10^{-x}$

4.10. $f(x) = \ln_2(-x)$ **4.11.** $f(x) = \ln(-x)$ **4.12.** $f(x) = \log(-x)$

Knowing forms of basic exponential and logarithmic functions (see P.4.1 to P.4.12), deduce and sketch graphs of functions in P.4.13 to P.4.17

4.13. $f(x) = 2^x - 2$ **4.14.** $f(x) = 2\ln(x) - 1$ **4.15.** $f(x) = 3\ \ln(-x) + 3$

4.16. $f(x) = 2^{\sqrt{x^2}}$ **4.17.** $f(x) = e^{\ln x}$ **4.18.** $f(x) = 3^x + 3^{-x}$

4.19. $f(x) = |\log_2 x|$ **4.20.** $f(x) = \left|5^{|x|} - 2\right|$ **4.21.** $f(x) = 3\ \ln(-x+2) + 3$

4.2 Simple Logarithmic Calculations

Without using calculator, calculate logarithms in P.4.22 to P.4.39

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023
R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41200-4_4

4.22. $\log_3 \frac{1}{81}$

4.23. $\log_{0.1} 1000$

4.24. $\log_2 \sqrt[3]{512}$

4.25. $\log_2 \frac{1}{128}$

4.26. $\log_{\sqrt{2}} 8$

4.27. $\log_a \sqrt[5]{a^2}$

4.28. $\log_{10} 10$

4.29. $\log_{10} 100$

4.30. $\log_{10} 1000$

4.31. $\log_2 16$

4.32. $\log_{25} \frac{1}{625}$

4.33. $\log_8 \frac{1}{4}$

4.34. $25^{\log_5 3}$

4.35. $\log_5 \sqrt{5}$

4.36. $\log_{2/3} \frac{243}{32}$

4.37. $\sqrt{\log^2_{1/2}(4)}$

4.38. $\sqrt{\log_{1/2}(4)}$

4.39. $\log_8 \left(\log_4 \left(\log_2 (16)\right)\right)$

4.3 Exponential Equations

Solve and comment on the solutions of equations in P.4.40 to P.4.49

4.40. $2^x = 8$

4.41. $2^{x-1} = 16$

4.42. $2^{x-5} = 3$

4.43. $e^{7-4x} = 6$

4.44. $e^{2x} - 3e^x + 2 = 0$

4.45. $5^{x+1} - 5^{x-1} = 24$

4.46. $\left(x^2 - x - 1\right)^{x^2-1} = 1$

4.47. $2(x+1)(3x+1)^x - (x-1) = (3x+1)^{x+1}$

4.48. $\left(\sqrt{5+\sqrt{24}}\right)^x + \left(\sqrt{5-\sqrt{24}}\right)^x = 10$

4.49. $81^x - 16^x - 2 \times 9^x(9^x - 4^x) + 36^x = 0$

4.4 Logarithmic Equations

Solve and comment on the solutions of equations in P.4.50 to P.4.55

4.50. $\log_2 x = 4$

4.51. $\ln(3x - 10) = 2$

4.52. $\ln(x^2 - 1) = 3$

4.53. $\ln x + \ln(x - 1) = 1$

4.54. $\ln(\ln x) = 1$

4.55. $\log_4(x + 1) = 1$

4.5 Exponential–Logarithmic Equations

Solve and comment on the solutions of equations in P.4.56 to P.4.58

4.56. $x^{\log_3 x} = 9$

4.57. $4^{(\log_4 x)^2} + x^{\log_4 x} = 512$

4.58. $5^{2(\log_5 2+x)} - 2 = 5^{x+\log_5 2}$

4.6 Exponential Inequalities

Solve and comment on the solutions of inequalities in P.4.59 to P.4.61

4.59. $2^{x+2} > \left(\frac{1}{4}\right)^{1/x}$

4.60. $1 < 3^{|x^2-x|} < 9$

4.61. $\sqrt{9^x - 3^{x+2}} > 3^x - 9$

4.7 Logarithmic Inequalities

Solve and comment on the solutions of inequalities in P.4.62 to P.4.64

4.62. $\log \frac{x-1}{x+2} > 0$

4.63. $\log_{2x+3} x^2 < 1$

4.64. $\log_{2x^2-x}(2x+2) < 1$

Answers

4.1 Logarithmic and Exponential Functions

Among all functions in mathematics, arguably, logarithmic and exponential forms are most challenging for beginners. Nevertheless, mastery of their basic forms and properties is the key for many other practical problems. In this section, exponential and logarithmic functions are reviewed from the visual perspective, in addition to some of the basic transformation techniques.

4.1. Regardless of its base, exponential functions cross vertical axis at $(x, y) = (0, 1)$ point. Note that exponential functions are always positive (see Fig. 4.1 (left)). In addition, at $x = 1$, exponential function equals to its base, i.e., 2, e, 10, etc. Exponential functions limit to "0" as $x \to -\infty$ and limit to $+\infty$ as $x \to \infty$.

4.2. Exponential function with base e has basically the same general form as the other exponential function, where its value at $x = 1$ equals to e. See Fig. 4.1 (left) where three typical exponential functions are compared.

4.3. Exponential function with base 10 has basically the same form as the other exponential function, where its value at $x = 1$ equals to 10. See Fig. 4.1 (left) where three typical exponential functions are compared.

4.4. Logarithmic function with the base 2 (see Fig. 4.1 (right)) has basically the same general form as logarithmic functions with some other base. Note that all basic logarithmic functions are defined for $x > 0$, they cross the horizontal axis at $(x, y) = (1, 0)$ point, and their respective value equals to "1" when x equals to their base, i.e., 2, e, 10, etc. Logarithmic function limit to $-\infty$ as $x \to 0$ from the positive side and limit to $+\infty$ as $x \to \infty$. Note rotational symmetry relative to exponential functions. Due to its relatively slow rise to infinity, in the communication theory, logarithmic functions are sometimes referred to as "compression" functions.

4.5. Logarithmic function with the base e (see Fig. 4.1 (right)) has basically the same general form as logarithmic functions with some other base.

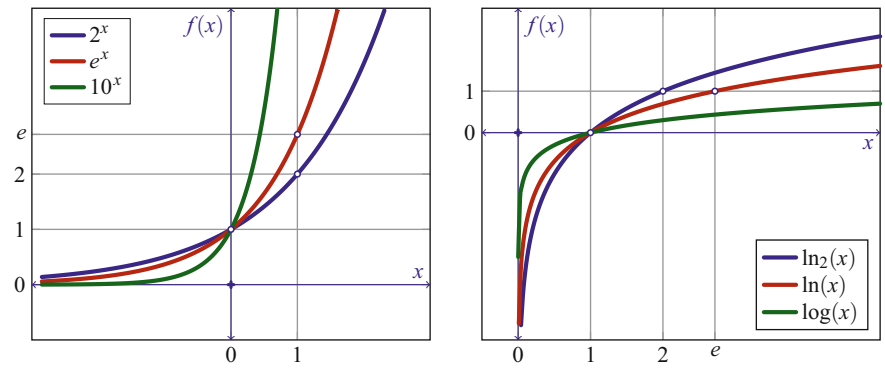


Fig. 4.1 Example P.4.1

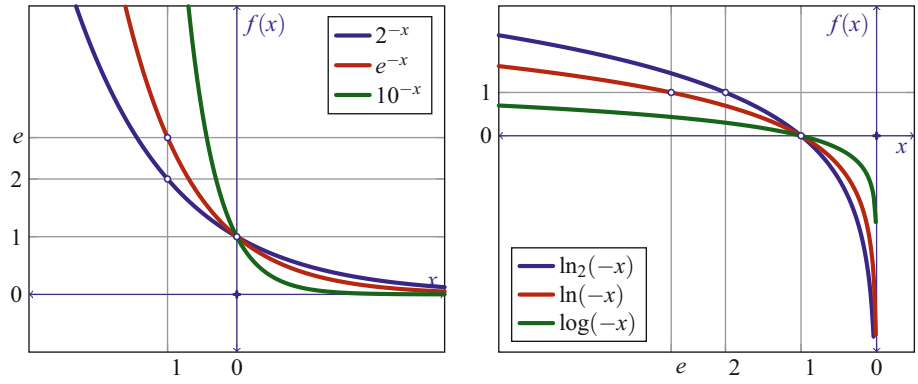


Fig. 4.2 Example P.4.7

4.6. Logarithmic function with the base 10 (see Fig. 4.1 (right)) has basically the same general form as logarithmic functions with some other base.

4.7. Regardless of its base, basic exponential functions with negative argument (see Fig. 4.2 (left)) are simple mirrored version of their respective form with the positive argument.

4.8. Regardless of its base, basic exponential functions with negative argument (see Fig. 4.2 (left)) are simple mirrored version of their respective form with the positive argument.

4.9. Regardless of its base, basic exponential functions with negative argument (see Fig. 4.2 (left)) are simple mirrored version of their respective form with the positive argument.

4.10. Regardless of its base, basic logarithmic functions with negative argument (see Fig. 4.2 (right)) are simple mirrored version of their respective form with the positive argument.

4.11. Regardless of its base, basic logarithmic functions with negative argument (see Fig. 4.2 (right)) are simple mirrored version of their respective form with the positive argument.

4.12. Regardless of its base, basic logarithmic functions with negative argument (see Fig. 4.2 (right)) are simple mirrored version of their respective form with the positive argument.

4.13. Starting from the basic exponential form 2^x, function $f(x) = 2^x - 2$ is found as 2^x simply shifted down along the vertical axis by "-2" (see Fig. 4.3 (left)).

4.14. Starting with the basic form of $\ln(x)$, first it is multiplied by "2" to create $2\ln(x)$ and then shifted by "-1" along the vertical axis to create $2\ln(x) - 1$ (see Fig. 4.3 (right)). For a simple sketch, it is sufficient to consider only a few typical points and to follow their movement. For example, $\ln(1) = 0$; therefore, $2\ln(1) = 0$, and $\ln(e) = 1$; therefore, $2\ln(e) = 2$. Then, $2\ln(e) - 1 = 1$, etc.

4.15. Starting with the basic form of $\ln(-x)$, first it is multiplied by "3" to create $3\ln(-x)$ and then shifted by "$+3$" along the vertical axis to create $3\ln(-x) + 3$ (see Fig. 4.4 (left)). For a simple sketch, it is sufficient to consider only a few typical points and to follow their movement. For example, $\ln(-1) = 0$; therefore, $3\ln(-1) = 0$, and $\ln(-e) = 1$; therefore, $3\ln(-e) = 3$. Then, $3\ln(-e) + 3 = 6$, $3\ln(-1) + 3 = 3$, etc.

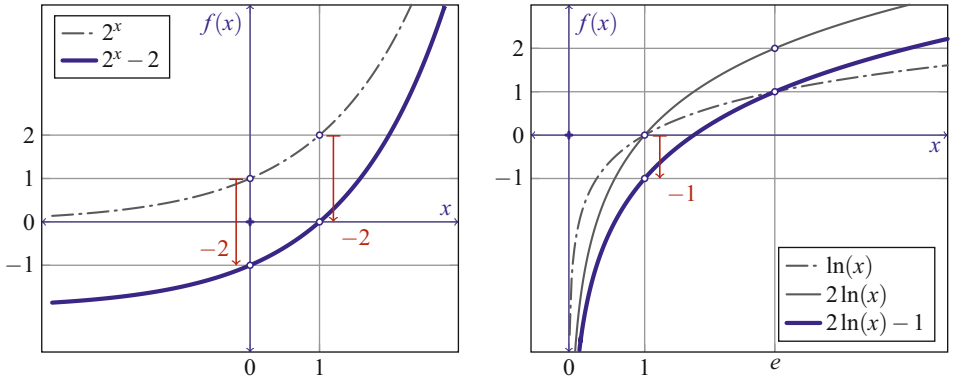


Fig. 4.3 Example P.4.13, P.4.14

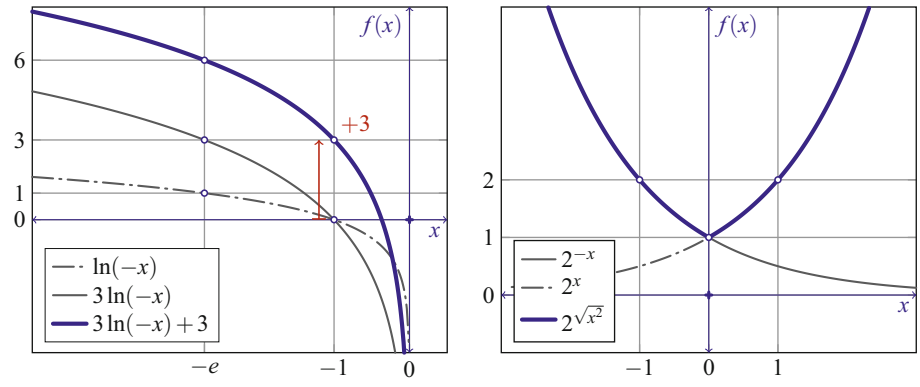


Fig. 4.4 Example P.4.15, P.4.16

4.16. Simple transformation shows that

$$f(x) = 2^{\sqrt{x^2}} = 2^{|x|} \quad \therefore \quad \begin{cases} f(x) = 2^x; & x \geq 0 \\ f(x) = 2^{-x}; & x \leq 0 \end{cases}$$

Therefore, function $f(x) = 2^{\sqrt{x^2}}$ is created by plotting only positive part of 2^x and negative part of 2^{-x} (see Fig. 4.4 (right)).

4.17. Knowing the basic function forms, first, argument of a logarithmic function must be strictly positive; thus, $x > 0$. Then, the relationship between exponential and logarithmic functions is that they are inverse to each other (similar like "+" and "−" or $\times$ and $\div$), that is,

$$f(x) = e^{\ln x} = x; \quad (x > 0)$$

In conclusion, this is a simple $f(x) = x$ function that is strictly defined only for $x > 0$ side.

Fig. 4.5 Example P.4.18

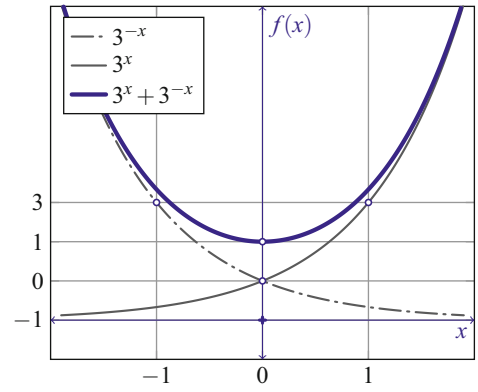
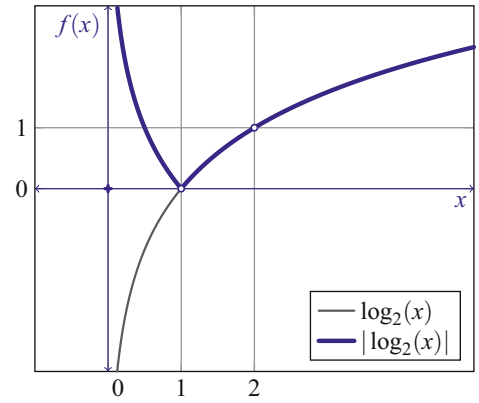


Fig. 4.6 Example P.4.19



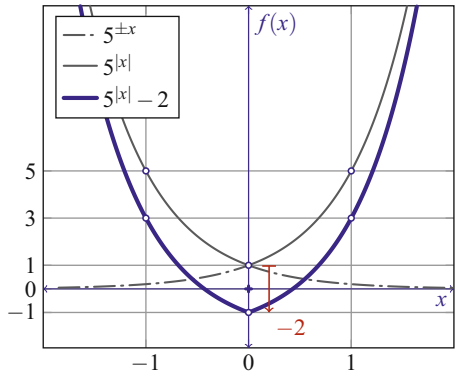
4.18. Starting with the basic functions 3^x and 3^{-x}, their sum may be sketched as in Fig. 4.5 For example, $3^0 = 1$, and $3^{-0} = 1$; thus, $3^0 + 3^{-0} = 2$, and the rest of the sum is sketched by following the asymptotic limits.

4.19. Starting from the basic form "$\log_2 x$" and knowing that absolute value of a positive arguments stays the same, while absolute value of a negative argument is multiplied by "-1" and becomes positive, then $|\log_2 x|$ is sketched as in Fig. 4.6

4.20. As already seen in P.4.16, exponent with the absolute argument may be created by combining exponent 5^x for $x \geq 0$ and exponent 5^{-x} for $x \leq 0$ to create $5^{|x|}$ (see Fig. 4.7 (left)). Then, the final form of $\left|5^{|x|} - 2\right|$ can be deduced by the following series of transformations:

$$5^{\pm x} \;\rightarrow\; 5^{|x|} \;\rightarrow\; 5^{|x|} - 2 \;\rightarrow\; \left|5^{|x|} - 2\right|$$

As the last transformation, the negative interval (shaded region) of $5^{|x|} - 2$ is simply mirrored to the positive side (see Fig. 4.7 (right)).



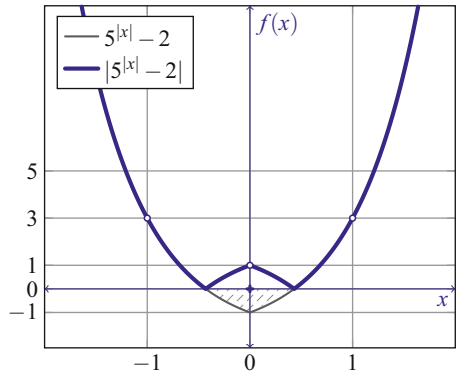
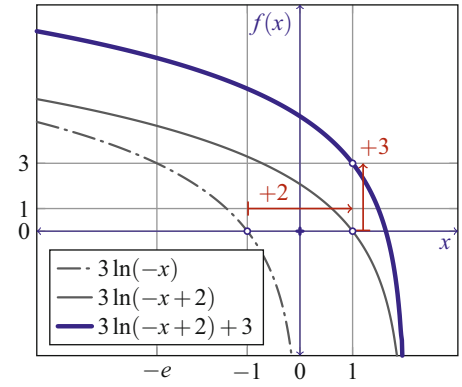


Fig. 4.7 Example P.4.20

Fig. 4.8 Example P.4.21



4.21. Starting from the basic function $\ln(-x)$, the final form of "$3\ \ln(-x+2)+3$" can be deduced by the following series of transformations:

$$\ln(-x) \ \rightarrow \ 3\ln(-x) \ \rightarrow \ 3\ln(-x+2) \ \rightarrow \ 3\ln(-x+2)+3$$

that appears as amplitude multiplied by "3"; then "$3\ln(-x)$" function is shifted to the right by "2" and moved up by "3" (see Fig. 4.8).

4.2 Simple Logarithmic Calculations

Problems in this section are intended for practicing hand calculations of logarithms with arbitrary base. The goal is to develop basic hand calculation skills needed for solving mathematical problems that include logarithmic functions.

Reminder: Arbitrary base logarithms are converted to rational functions of $\ln(x)$ as

$$\log_a b = \frac{\ln b}{\ln a}$$

4.22. Given $x = \log_3(1/81)$, argument of an logarithm with the arbitrary base may be resolved by using the powers of the same base as

$$x = \log_3 \frac{1}{81} \;\;\therefore\;\; 3^x = 3^{\log_3(1/81)} \;\;\therefore\;\; 3^x = \frac{1}{81} = \frac{1}{3^4} = 3^{-4} \;\;\therefore\;\; x = \log_3 \frac{1}{81} = -4$$

which is result of a simple deduction: if $3^x = 3^{-4}$, then it must be that $x = -4$ (because of the same power base).

4.23. Note that the logarithm's base is 0.1, and by following the same reasoning as in P.4.22, it follows that

$$x = \log_{0.1} 1000 \;\;\therefore\;\; 0.1^x = 1000 = 10^3 = \frac{1}{10^{-3}} = 0.1^{-3} \;\;\therefore\;\; x = \log_{0.1} 1000 = -3$$

4.24. $x = \log_2 \sqrt[3]{512} \;\;\therefore\;\; 2^x = \sqrt[3]{512} = \left(2^9\right)^{1/3} = 2^3 \;\;\therefore\;\; x = \log_2 \sqrt[3]{512} = 3$

4.25. $x = \log_2 \frac{1}{128} \;\;\therefore\;\; 2^x = 2^{\log_2(1/128)} \;\;\therefore\;\; 2^x = \frac{1}{128} = \frac{1}{2^7} = 2^{-7} \;\;\therefore\;\; x = \log_2 \frac{1}{128} = -7$

4.26. $x = \log_{\sqrt{2}} 8 \;\;\therefore\;\; \left(\sqrt{2}\right)^x = \sqrt{2}^{\log_{\sqrt{2}} 8} = 8 = \left(\sqrt{2}\right)^6 \;\;\therefore\;\; x = \log_{\sqrt{2}} 8 = 6$

4.27. $x = \log_a \sqrt[5]{a^2} \;\;\therefore\;\; a^x = \sqrt[5]{a^2} = a^{2/5} \;\;\therefore\;\; x = \log_a \sqrt[5]{a^2} = \frac{2}{5}$

4.28. $x = \log_{10} 10 \;\;\therefore\;\; 10^x = 10 \;\;\therefore\;\; x = 1$

4.29. $x = \log_{10} 100 \;\;\therefore\;\; 10^x = 100 \;\;\therefore\;\; x = 2$

4.30. $x = \log_{10} 1000 \;\;\therefore\;\; 10^x = 1000 \;\;\therefore\;\; x = 3$

In addition, writing "log" without specific index number assumes the base 10, while writing "ln" assumes the base e. Note that, for example, $x = \log 1{,}000{,}000$ may be found by simply counting the zeros; therefore $x = 6$, $x = \log 1000$; therefore $x = 3$, etc.

4.31. $x = \log_2 16 \;\;\therefore\;\; 2^x = 16 \;\;\therefore\;\; 2^x = 2^4 \;\;\therefore\;\; x = \log_2 16 = 4$

4.32. $x = \log_{25} \frac{1}{625} = \log_{25} \frac{1}{25^2} = \log_{25} 25^{-2} = -2 \log_{25} 25^{1} \;\;\therefore\;\; x = \log_{25} \frac{1}{625} = -2$
where logarithm of its base always equals to "1" and identity "$\log a^b = b \log a$" is used to simplify powers of logarithm's argument.

4.33. $\log_8 \frac{1}{4} = \log_8 \frac{1}{\sqrt[3]{64}} = \log_8 \frac{1}{\sqrt[3]{8^2}} = \log_8 8^{-2/3} = -\frac{2}{3} \log_8 8^{1} \;\;\therefore\;\; \log_8 \frac{1}{4} = -\frac{2}{3}$

4.34. $25^{\log_5 3} = \left(5^2\right)^{\log_5 3} = 5^{(2 \log_5 3)} = 5^{\log_5 3^2} \;\;\therefore\;\; 25^{\log_5 3} = 9$

4.35. $\log_5 \sqrt{5} = \log_5 5^{1/2} = \frac{1}{2} \log_5 5^{1} \;\;\therefore\;\; \log_5 \sqrt{5} = \frac{1}{2}$

4.36. $\log_{2/3}\dfrac{243}{32} = \log_{2/3}\dfrac{3^5}{2^5} = \log_{2/3}\left(\dfrac{2}{3}\right)^{-5} = -5\log_{2/3}\dfrac{2}{3}^{1} \;\therefore\; \log_{2/3}\dfrac{243}{32} = -5$

4.37. $\sqrt{\log^2_{1/2}(4)} = \left|\log_{1/2}(4)\right| = \left|\log_{1/2}\left(\dfrac{1}{2}\right)^{-2}\right| = \left|-2\log_{1/2}\dfrac{1}{2}^{1}\right| = |-2| \;\therefore\; \sqrt{\log^2_{1/2}(4)} = 2$

4.38. $\sqrt{\log_{1/2}(4)} = \sqrt{\log_{1/2}\left(\dfrac{1}{2}\right)^{-2}} = \sqrt{-2\log_{1/2}\dfrac{1}{2}^{1}} = \sqrt{-2} \;\therefore\; \sqrt{\log_{1/2}(4)} = j\sqrt{2}$

4.39. Given a general composed "telescopic" form $x = \log_8\left(\log_4\left(\log_2(16)\right)\right)$, start unfolding it from the inside (as the "Russian doll"), as

$$x = \log_8\left(\log_4\left(\log_2(16)\right)\right) = \log_8\left(\log_4\left(\log_2\left(2^4\right)\right)\right) = \log_8\left(\log_4\left(4\log_2(2)^{1}\right)\right)$$
$$= \log_8\left(\log_4 4^{1}\right) \Rightarrow x = \log_8(1)$$
$$\therefore\; 8^x = 8^{\log_8(1)} \Rightarrow 8^x = 1 = 8^0 \;\therefore\; x = 0$$
$$\therefore$$
$$\log_8\left(\log_4\left(\log_2(16)\right)\right) = 0$$

4.3 Exponential Equations

Exponentials of an arbitrary base are often resolved with the help of logarithm with the same base.

4.40. $2^x = 8 \;\therefore\; 2^x = 2^3 \;\therefore\; x = 3$

4.41. $2^{x-1} = 16 \;\therefore\; 2^{x-1} = 2^4 \;\therefore\; x - 1 = 4 \;\therefore\; x = 5$

4.42. $2^{x-5} = 3 \;\therefore\; \log_2\left(2^{x-5}\right) = \log_2 3 \;\therefore\; x - 5 = \log_2 3 = \dfrac{\ln 3}{\ln 2} \;\therefore\; x = \dfrac{5\ln 2 + \ln 3}{\ln 2}$

4.43. $e^{7-4x} = 6 \;\therefore\; \ln\left(e^{7-4x}\right) = \ln 6 \;\therefore\; 7 - 4x = \ln 6 \;\therefore\; 7 - \ln 6 = 4x \;\therefore\; x = \dfrac{7 - \ln 6}{4}$

4.44. This exponential equation is in reality quadratic equation in disguise; by the change of variables technique, it is converted into a simple quadratic equation as

$$e^{2x} - 3e^x + 2 = 0 \;\therefore\; \left(e^x\right)^2 - 3\left(e^x\right) + 2 = 0 \;\therefore\; t = e^x$$
$$\therefore\; t^2 - 3t + 2 = 0$$
$$t^2 - 2t - t + 2 = 0$$

$$t(t-2)-(t-2)=0$$
$$(t-2)(t-1)=0 \;\;\therefore\;\; \underline{t_1=2,\;\; t_2=1}$$

$\underline{\textit{Case } t_1=2:}\;\; \therefore\;\; e^{x_1}=2 \;\;\therefore\;\; \ln\left(e^{x_1}\right)=\ln 2 \;\;\therefore\;\; \underline{x_1=\ln 2}$
$\underline{\textit{Case } t_2=2:}\;\; \therefore\;\; e^{x_2}=1 \;\;\therefore\;\; \ln\left(e^{x_2}\right)=\ln 1 \;\;\therefore\;\; \underline{x_1=0}$

4.45. After factorization of 5^2 term, it follows that

$$5^{x+1}-5^{x-1}=24$$
$$5^{x-1}5^2-5^{x-1}=24$$
$$5^{x-1}\left(5^2-1\right)=24$$
$$5^{x-1}\times 24=24 \;\;\therefore\;\; 5^{x-1}=1=5^0 \;\;\therefore\;\; x-1=0 \;\;\therefore\;\; x=1$$

4.46. Given $f(x)=\left(x^2-x-1\right)^{x^2-1}=1$, note that the general form of this function is $a^b=1$, which is equivalent to logarithmic form

$$\ln a^b=\ln 1 \;\;\therefore\;\; b\,\ln a=0$$

where argument of a logarithmic function must be strictly positive, that is, $a>0$. In this problem, it must be that $x^2-x-1>0$. Roots of this polynomial are

$$x^2-x-1=0 \;\;\therefore\;\; x_1=\frac{1-\sqrt{5}}{2} \quad \text{and} \quad x_2=\frac{1+\sqrt{5}}{2}$$

where inequality $x^2-x-1>0$ defines two intervals: $x<x_1$ and $x>x_2$.

By consequence, in this problem, equation $a^b=1$ is possible in two cases: 1) if $a>0$ and $b=0$, or 2) $a=1$. Thus,

$\underline{\textit{Case 1 :}}\;\; a^b=1\Rightarrow a>0$ and $b=0$, that is to say

$$\left(x^2-x-1\right)^{x^2-1}=1 \;\;\therefore\;\; \begin{cases} x^2-x-1 & >0 \;\;\therefore\;\; (x^2-1)-x>0 \\ x^2-1 & =0 \;\;\therefore\;\; x=\pm 1 \end{cases}$$

which is to say,

$\underline{x=1}\;:\;\; \Rightarrow x^2-1=0 \;\;\therefore\;\; (x^2-1)-x=0-1>0 \;\;\therefore\;\; -1>0$ ✗
$\underline{x=-1}\;:\;\; \Rightarrow x^2-1=0 \;\;\therefore\;\; (x^2-1)-x=0-1=0-(-1)\neq 0 \;\;\therefore\;\; 1>0$ ✓

therefore, $x=-1$ satisfies both conditions.

$\underline{\textit{Case 2 :}}\;\; a^b=1\Rightarrow a=0$, thus

$$x^2-x-1=1 \;\;\therefore\;\; x^2-x-2=0 \;\;\therefore\;\; x^2-2x+x-2=0 \;\;\therefore\;\; x(x-2)+x-2=0$$

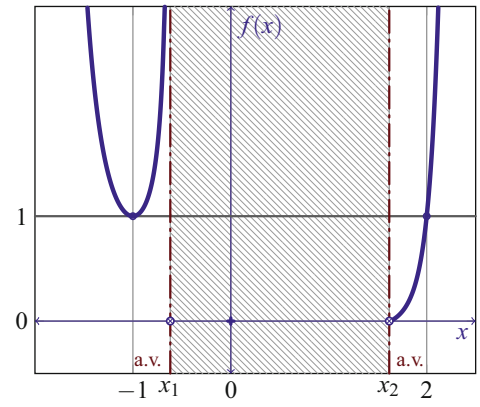


Fig. 4.9 Example P.4.46

$$\therefore \quad (x+1)(x-2)=0 \quad \therefore \quad x=-1, x=2$$

Thus, in total $x=-1$ (double root) and $x=2$. These three solutions for $f(x)=1$ are illustrated in Fig. 4.9

4.47. Given equation $2(x+1)(3x+1)^x-(x-1)=(3x+1)^{x+1}$ note that it must be (see P.4.46)

$$3x+1>0 \quad \therefore \quad x>\frac{1}{3}$$

then, this function may be converted into the following form:

$$\begin{aligned}
2(x+1)(3x+1)^x-(x-1)&=(3x+1)^{x+1}\\
2(x+1)(3x+1)^x-(3x+1)^{x+1}&=x-1\\
2(x+1)(3x+1)^x-(3x+1)(3x+1)^x&=x-1\\
(3x+1)^x\big[2(x+1)-(3x+1)\big]&=x-1\\
-(x-1)(3x+1)^x&=x-1
\end{aligned}$$

Note that $(3x+1)^x>0$ for all $x>1/3$, and $-(x-1)\neq(x-1)$; thus, the last equation is possible only if $x=1$, i.e.,

$$\underline{x=1} \ : \quad -(1-1)(3(1)+1)^1=1-1 \quad \therefore \quad -(0)(4)=0 \ \checkmark$$

In conclusion, $x=1$. This solution may be illustrated by finding the intersection point between the left and right sides of equation (see Fig. 4.10), i.e.

$$\begin{aligned}
f_1(x)&=2(x+1)(3x+1)^x-(x-1)\\
f_2(x)&=(3x+1)^{x+1}
\end{aligned}$$

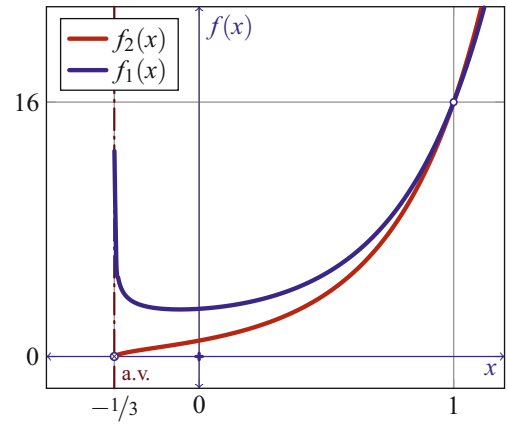


Fig. 4.10 Example P.4.47

4.48. By using the change of variables technique, given equation is converted as

$$\left(\sqrt{5+\sqrt{24}}\right)^x + \left(\sqrt{5-\sqrt{24}}\right)^x = 10$$

$$\left(\sqrt{5+\sqrt{24}}\ \frac{\sqrt{5-\sqrt{24}}}{\sqrt{5-\sqrt{24}}}\right)^x + \left(\sqrt{5-\sqrt{24}}\right)^x = 10$$

$$\left(\frac{\sqrt{(5+\sqrt{24})(5-\sqrt{24})}}{\sqrt{5-\sqrt{24}}}\right)^x + \left(\sqrt{5-\sqrt{24}}\right)^x = 10$$

$$\left(\frac{\sqrt{25-24}}{\sqrt{5-\sqrt{24}}}\right)^x + \left(\sqrt{5-\sqrt{24}}\right)^x = 10$$

$$\frac{1}{\left(\sqrt{5-\sqrt{24}}\right)^x} + \left(\sqrt{5-\sqrt{24}}\right)^x = 10$$

This form is converted into quadratic equation with $\left(\sqrt{5-\sqrt{24}}\right)^x = t$ change, as

$$\frac{1}{t} + t = 10 \ \ \therefore \ \ t^2 - 10\,t + 1 = 0 \ \ \therefore \ \ t_{1,2} = 5 \pm \sqrt{24}$$

Return to the original variable results in

$$\left(\sqrt{5-\sqrt{24}}\right)^x = t \ \ \therefore \ \ \left(\sqrt{5-\sqrt{24}}\right)^x = 5-\sqrt{24} \ \ \therefore \ \ \underline{x=2}$$

$$\therefore \ \ \left(\sqrt{5-\sqrt{24}}\right)^x = 5+\sqrt{24} = \frac{1}{5-\sqrt{24}} \ \ \therefore \ \ \underline{x=-2}$$

$$\left\{x^{-n} = \frac{1}{x^n};\ \ (a-b)(a+b) = a^2 - b^2\right\}$$

4.49. After factorization, this equitation is transformed into quadratic equation as

$$\begin{aligned}
81^x - 16^x - 2 \times 9^x (9^x - 4^x) + 36^x &= 0 \\
(3^4)^x - (2^4)^x - 2 \times 9^{2x} + 2 \times 9^x 4^x) + (6^2)^x &= 0 \\
3^{4x} - 2^{4x} - 2 \times 3^{4x} + 2 \times 3^{2x} 2^{2x} + 6^{2x} &= 0 \\
3^{4x} - 2 \times 3^{4x} - 2^{4x} + 2 \times 6^{2x} + 6^{2x} &= 0 \\
-3^{4x} - 2^{4x} + 3 \times 6^{2x} &= 0 \\
3^{4x} + 2^{4x} - 3 \times 6^{2x} &= 0 \quad \{\text{factor } 2^{4x}\} \\
\frac{3^{4x}}{2^{4x}} + \frac{\cancel{2^{4x}}}{\cancel{2^{4x}}} - 3\,\frac{3^{2x}\,\cancel{2^{2x}}}{\cancel{2^{4x}}\,\cancel{2^{2x}}} &= 0 \\
\left[\left(\frac{3}{2}\right)^{2x}\right]^2 - 3\left(\frac{3}{2}\right)^{2x} + 1 &= 0
\end{aligned}$$

Thus, change of variable leads to

$$t = \left(\frac{3}{2}\right)^{2x} \quad \therefore \quad t^2 - 3t + 1 = 0 \quad \therefore \quad t_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

Return to the original variable gives

$$\left(\frac{3}{2}\right)^{2x} = \left(\frac{9}{4}\right)^{x} = \frac{3 \pm \sqrt{5}}{2} \quad \therefore \quad x_{1,2} \ln\frac{9}{4} = \ln\frac{3 \pm \sqrt{5}}{2} \quad \therefore \quad x_{1,2} = \frac{\ln(3 \pm \sqrt{5}) - \ln 2}{\ln 9 - \ln 4}$$

4.4 Logarithmic Equations

Logarithmic and exponential functions reverse each other, similar as add/subtract or multiply/divide do; thus, the general strategy for solving these types of equations is to apply opposite function to both left and right sides of equation.

Reminder: Given a logarithmic equation, its solution is found by rising both left and right side of the equation as

$$\log_a f(x) = b \Rightarrow a^{\log_a f(x)} = a^b \Rightarrow f(x) = a^b$$

or, given exponential function

$$a^{f(x)} = b \Rightarrow \log_a a^{f(x)} = \log_a b \Rightarrow f(x) \underbrace{\log_a a}_{1} = \log_a b \Rightarrow f(x) = \log_a b$$

4.50. $\log_2 x = 4 \;\;\therefore\;\; 2^{\log_2 x} = 2^4 \;\;\therefore\;\; x = 2^4 = 16$

4.51. $\ln(3x-10) = 2 \;\;\therefore\;\; e^{\ln(3x-10)} = e^2 \;\;\therefore\;\; 3x - 10 = e^2 \;\;\therefore\;\; x = \dfrac{e^2+10}{3}$

4.52. $\ln(x^2-1) = 3 \;\;\therefore\;\; e^{\ln(x^2-1)} = e^3 \;\;\therefore\;\; x^2 - 1 = e^3 \;\;\therefore\;\; x = \pm\sqrt{e^3+1}$

4.53. $\ln x + \ln(x-1) = 1 \;\;\therefore\;\; \ln\big(x(x-1)\big) = 1 \;\;\therefore\;\; e^{\ln(x(x-1))} = e \;\;\therefore\;\; x^2 - x = e$

$$x^2 - x - e = 1 \;\;\therefore\;\; x_{1,2} = \frac{1 \pm \sqrt{1+4e}}{2}$$

4.54. $\ln(\ln x) = 1 \;\;\therefore\;\; e^{\ln(\ln x)} = e \;\;\therefore\;\; \ln x = e \;\;\therefore\;\; e^{\ln x} = e^e \;\;\therefore\;\; x = e^e$

4.55. $\log_4(x+1) = 1 \;\;\therefore\;\; 4^{\log_4(x+1)} = 4^1 \;\;\therefore\;\; x + 1 = 4 \;\;\therefore\;\; x = 3$

4.5 Exponential–Logarithmic Equations

Equations that contain various composite exp/log forms are resolved with successive use of identities that gradually simplify the original form.

4.56. By using the identities for log and exp functions, it follows that

$$x^{\log_3 x} = 9 \;\;\therefore\;\; \log_3\left(x^{\log_3 x}\right) = \log_3 9 \;\;\therefore\;\; \log_3\left(x^{\log_3 x}\right) = \log_3 3^2 \;\;\therefore\;\; \log_3 x \;\log_3 x = 2\log_3 3^{\to 1}$$
$$\therefore\;\; \log_3 x \log_3 x = 2 \;\;\therefore\;\; \left(\log_3 x\right)^2 = 2 \;\;\therefore\;\; \log_3 x = \pm\sqrt{2} \;\;\therefore\;\; 3^{\log_3 x} = 3^{\pm\sqrt{2}}$$
$$\therefore\;\; x = 3^{\pm\sqrt{2}}$$

4.57. By exploiting exponential/logarithmic identities, it follows that

$$4^{(\log_4 x)^2} + x^{\log_4 x} = 512 \quad \left\{a^{b^2} = a^{b\,b} = (a^b)^b\right\}$$
$$\left(4^{\log_4 x}\right)^{\log_4 x} + x^{\log_4 x} = 512$$
$$x\,\log_4 x + x^{\log_4 x} = 512$$
$$2\,x^{\log_4 x} = 512$$
$$x^{\log_4 x} = 256$$
$$\log_4\left(x^{\log_4 x}\right) = \log_4 256 \quad \left\{\log a^b = b\log a,\;\; 256 = 4^4\right\}$$
$$\log_4 x \;\log_4 x = \log_4 4^4 = 4 \;\;\therefore\;\; \left(\log_4 x\right)^2 = 4 \;\;\therefore\;\; \log_4 x = \pm 2$$
$$4^{\log_4 x} = 4^{\pm 2} \;\;\therefore\;\; x = 4^{\pm 2}$$
$$x_1 = 4^2 = 16, \quad x_2 = 4^{-2} = \frac{1}{16}$$

4.58. As a consequence of exponential and logarithmic identities, any real number can be written as exponent of an arbitrary base logarithm, e.g., $2 = 5^{\log_5 2}$; thus

$$5^{2(\log_5 2+x)} - 2 = 5^{x+\log_5 2}$$

$$\left\{a^{b+c} = a^b\, a^c;\quad a = n^{\log_n a};\quad x^{ab} = \left(x^a\right)^b\right\}$$

$$5^{2\log_5 2}\, 5^{2x} - 5^{\log_5 2} = 5^x\, 5^{\log_5 2}$$

$$\left(5^{\log_5 2}\right)^2\, 5^{2x} - 5^{\log_5 2} - 5^x\, 5^{\log_5 2} = 0$$

$$5^{\log_5 2}\left(5^{\log_5 2}\, 5^{2x} - 1 - 5^x\right) = 0$$

$$2\left(5^{\log_5 2}\, 5^{2x} - 1 - 5^x\right) = 0$$

$$5^{\log_5 2}\, 5^{2x} - 5^x - 1 = 0$$

$$2 \times 5^{2x} - 5^x - 1 = 0$$

By the change of variables technique, the last equation is converted into a quadratic equation.

$$2 \times \left(5^x\right)^2 - 5^x - 1 = 0 \quad \left\{5^x = t\right\}$$

$$2t^2 - t - 1 = 0$$

$$2t^2 - 2t + t - 1 = 0$$

$$2t(t-1) + (t-1) = 0$$

$$(t-1)(2t+1) = 0 \quad \therefore$$

$$t_1 = 1 \quad \text{or} \quad t_2 = -\frac{1}{2} \quad \therefore$$

$$5^x = -\frac{1}{2} \quad \text{not possible because} \quad 5^x > 0 \ \ \forall x$$

$$5^x = 1 \quad \therefore \quad \underline{x = 0}$$

This solution may be illustrated by finding the intersection point between the left and right sides of equation (see Fig. 4.11), i.e.,

$$f_1(x) = 5^{2(\log_5 2+x)} - 2$$

$$f_2(x) = 5^{x+\log_5 2}$$

4.6 Exponential Inequalities

Similar to their equation counterparts, exponential inequalities result in non-unique answers. Instead, the solution set consists of one or multiple intervals. In general, exponential forms are considered easier to solve, because basic exponential functions are well-defined and positive.

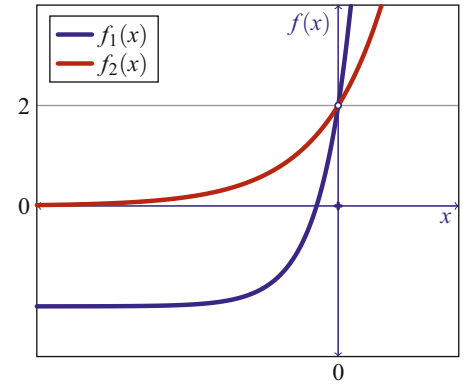


Fig. 4.11 Example P.4.58

4.59. The main idea is transform the inequality terms until it is possible to compare powers of the same base, i.e.,

$$2^{x+2} > \left(\frac{1}{4}\right)^{1/x} \quad \therefore \quad 2^{x+2} > 4^{-1/x} \quad \therefore \quad 2^{x+2} > \left(2^2\right)^{-1/x} \quad \therefore \quad 2^{x+2} > 2^{-2/x}$$

Therefore, this inequality is equivalent to

$$x + 2 > -\frac{2}{x} \quad \therefore \quad x + 2 + \frac{2}{x} > 0$$

Note that for this inequality $x \neq 0$, due to division by zero in inverse function "$2/x$," it is necessary to examine two intervals:

1. $x > 0:$ Obviously, within this interval, the inequality is always satisfied because all three terms on the left side are positive. Therefore, one solution is the interval $x > 0$.
2. $x < 0:$ Within this interval, the two terms that include x are obviously negative, while "$+2$" term is always positive. Thus

$$x + 2 + \frac{2}{x} > 0 \quad \therefore \quad \frac{x^2 + 2x + 2}{x} > 0$$

Note that, given $x < 0$ interval, numerator "$x^2 + 2x + 2$" is always positive, while denominator "x" is always negative. Consequently, rational term on the left side of inequality is always negative, not positive.

In conclusion, solution to this inequality is "$x > 0$" interval.

4.60. Given double strict inequality with the absolute expression $|x^2 - x| = |x(x - 1)|$, there are two expressions to be considered.

$$|x(x-1)|=\begin{cases} x(x-1)>0 \;\;\therefore\;\; x>0 \;\text{ and }\; x-1>0 \;\therefore\; x>1 \;\Rightarrow |x^2-x|=x^2-x \\ \qquad\qquad\quad x<0 \;\text{ and }\; x-1<0 \;\therefore\; x<0 \;\Rightarrow |x^2-x|=x^2-x \\ x^2-x<0 \;\;\therefore\;\; x>0 \;\text{ and }\; x-1<0 \;\therefore\; 0<x<1 \;\Rightarrow |x^2-x|=-(x^2-x) \\ \qquad\qquad\quad x<0 \;\text{ and }\; x-1>0 \;\Rightarrow \text{(empty interval)} \end{cases}$$

Therefore,

1. $x>1$ or $x<0$: $1<3^{x^2-x}<9$

$$1<3^{x^2-x} \;\therefore\; \log_3 1<\log_3 3^{x^2-x} \;\therefore\; 0<x^2-x \Rightarrow x>1 \quad\text{or}\quad x<0$$

$$3^{x^2-x}<9 \;\therefore\; 3^{x^2-x}<3^2 \;\therefore\; x^2-x<2 \;\therefore\; x^2-x-2<0 \;\therefore\; x^2-2x+x-2<0$$

$$\therefore\; x(x-2)+x-2<0$$

$$(x-2)(x+1)<0 \Rightarrow x>-1 \quad\text{and}\quad x<2$$

$$x<-1 \quad\text{and}\quad x>2 \quad\text{(empty interval)}$$

2. $0<x<1$: $1<3^{-(x^2-x)}<9$

$$1<3^{-(x^2-x)} \;\therefore\; \log_3 1<\log_3 3^{-(x^2-x)} \;\therefore\; 0<-(x^2-x) \;\therefore\; x^2-x<0 \Rightarrow 0<x<1$$

$$3^{-(x^2-x)}<9 \;\therefore\; 3^{-(x^2-x)}<3^2 \;\therefore\; x-x^2<2 \;\therefore\; -x^2+x-2<0 \;\checkmark\; (x\in\forall x)$$

In summary, the solution set is in interval (note strict inequalities)

$$\left.\begin{array}{l} -1<x<2 \\ x>1 \quad\text{or}\quad x<0 \\ 0<x<1 \end{array}\right\} \;\therefore\; x\in(-1,2) \quad\text{interval, where,}\quad x\neq(-1,0,1,2)$$

This solution set is illustrated by non-shaded region in Fig. 4.12 (left). As a comparison, the rational term itself that is bounded by "1" and "9" is shown in Fig. 4.12 (right), where points excluded due to strict inequalities are crossed.

4.61. Following the idea of reducing all terms to the same base, we write

$$\sqrt{9^x-3^{x+2}}>3^x-9$$

$$\sqrt{3^{2x}-3^x3^2}>3^x-3^2 \qquad \Big|^2$$

$$3^{2x}-3^x3^2>3^{2x}-2\times3^x3^2+3^4$$

$$2\times3^x3^2-3^x3^2>3^4$$

$$3^x3^2>3^4 \;\therefore\; x>2$$

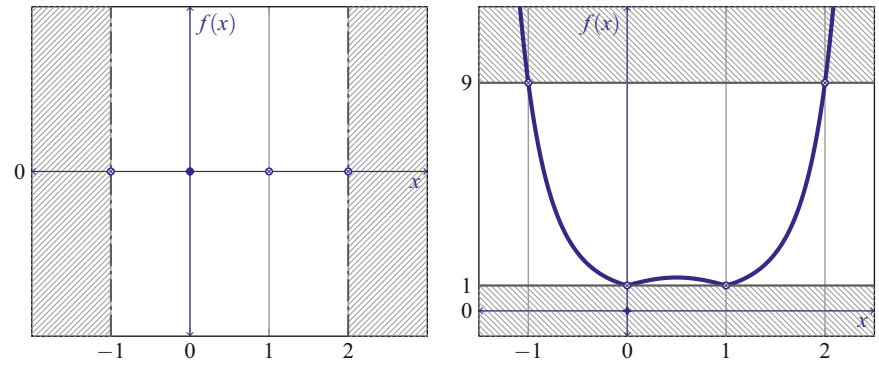


Fig. 4.12 Example P.4.60

4.7 Logarithmic Inequalities

Similar to their equation counterparts, logarithmic inequalities result in non-unique answers, and the solution set consists of one or multiple intervals. In general, solving logarithmic forms is a bit more involved due to constrains of basic logarithmic functions.

4.62. Logarithmic function is positive when its argument is superior to one, i.e.,

$$\log \frac{x-1}{x+2} > 0 \quad \therefore \quad \frac{x-1}{x+2} > 1 \quad \text{where,} \quad x \neq -2$$

The solution set may be deduced as follows:

$$\frac{x-1}{x+2} > 1 \quad \therefore \quad \frac{x-1}{x+2} - 1 > 0 \quad \therefore \quad \frac{\cancel{x}-1-\cancel{x}-2}{x+2} > 0 \quad \therefore \quad \frac{-3}{x+2} > 0 \quad \therefore \quad x+2 < 0$$
$$\therefore \quad \underline{x < -2}$$

Therefore, $x < -2$ is the solution set.

4.63. Given inequality

$$\log_{2x+3} x^2 < 1 \quad \therefore \quad \frac{\ln x^2}{\ln(2x+3)} < 1$$

Note that logarithm's argument must be strictly positive; thus, $x \neq 0$. It is important to distinguish logarithms whose base is inferior to "1" from those whose base is superior to "1." Note that two logarithmic functions with the inverse bases are symmetric relative to the horizontal axis. Also, note that "1" can be written as an arbitrary base logarithm of that base, for example,

$$1 = \log_{2x+3}(2x+3)$$

Case 1: $0 < 2x+3 < 1$

$$\left.\begin{array}{ll} 0 < 2x+3 & \therefore\ x > -\dfrac{3}{2} \\ 2x+3 < 1 & \therefore\ x < -1 \end{array}\right\} \Rightarrow \underline{-\frac{3}{2} < x < 1}$$

Then,

$$\log_{2x+3}\ x^2 < 1$$

$$\log_{2x+3}\ x^2 < \log_{2x+3}(2x+3)$$

$$\therefore\ x^2 > 2x+3\ \ \therefore\ x^2 - 2x - 3 > 0\ \ \therefore\ x^2 - 3x + x - 3 > 0$$

$$\therefore\ x(x-3) + x - 3 > 0\ \ \therefore\ (x+1)(x-3) > 0$$

$$(x+1) > 0 \quad \text{and} \quad (x-3) > 0\ \ \therefore\ x > 3\ \text{✗}\ (\text{note}, x < 1)$$

or,

$$(x+1) < 0 \quad \text{and} \quad (x-3) < 0\ \ \therefore\ \underline{x < -1}$$

Case 2: $2x+3 > 1\ \ \therefore\ x > -1$, then

$$\log_{2x+3}\ x^2 < 1$$

$$\log_{2x+3}\ x^2 < \log_{2x+3}(2x+3)\ \ \therefore\ x^2 < 2x+3\ \ \therefore\ x^2 - 2x - 3 < 0$$

$$\therefore$$

$$(x+1) > 0 \quad \text{and} \quad (x-3) < 0\ \ \therefore\ \underline{-1 < x < 3}$$

or,

$$(x+1) < 0 \quad \text{and} \quad (x-3) > 0\ \ (\text{empty interval})$$

Therefore, the complete solution set is $x \in (-3/2, 3)$ interval, where points that are *not* included due to strict inequalities are $(x \neq -3/2, -1, 0, 3)$. This solution set is illustrated by non-shaded region in Fig. 4.13 (left). As a comparison, the arbitrary base logarithm term itself that is bounded by "1" is shown in Fig. 4.13 (right), where points excluded due to strict inequalities are crossed.

4.64. Logarithm function of an arbitrary base may be evaluated by converting given function into ratio of two logarithmic functions as

$$\log_{2x^2-x}(2x+2) < 1 \quad \left\{\log_a b = \frac{\ln b}{\ln a}\right\} \quad \therefore\ \frac{\ln(2x+2)}{\ln(2x^2-x)} < 1$$

Basic properties of $\log(x)$ function are as follows: $x > 0$, if $(0 < x < 1)$, $\log(1) = 0$, then $\log x < 0$, and if $(x > 1)$, then $\log x > 0$. Thus,

1. Rational function is not defined when denominator equals to zero, i.e., the result of division is either infinity or undetermined; thus

$$\ln(2x^2 - x) \neq 0\ \ \therefore\ 2x^2 - x \neq 1\ \ \therefore\ 2x^2 - x - 1 \neq 0$$

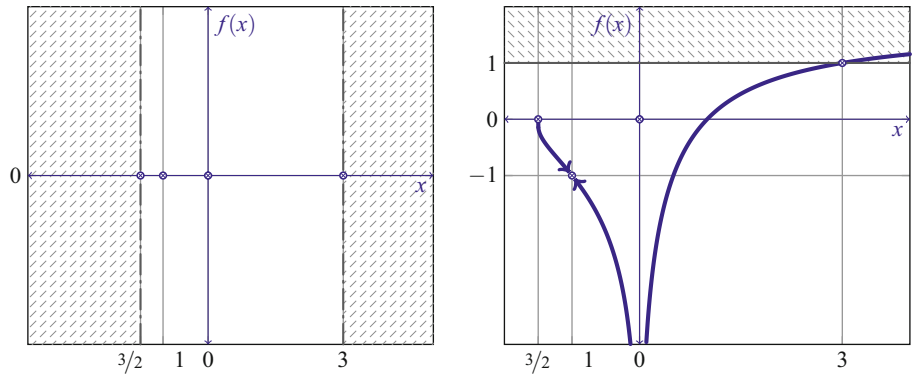


Fig. 4.13 Example P.4.63

$$\therefore \quad x \neq 1, x \neq -\frac{1}{2}$$

$$\ln(2x^2 - x) < 0 \quad \therefore \quad 0 < x(2x-1) < 1 \quad \therefore \quad 0 < x < \frac{1}{2}$$

and this logarithm function is defined for positive argument, i.e.,

$$2x^2 - x > 0 \quad \therefore \quad x(2x-1) > 0 \quad \therefore \quad x < 0 \quad \text{or,} \quad x > \frac{1}{2}$$

2. Similarly, logarithm in the numerator is defined for

$$2x + 2 > 0 \quad \therefore \quad x > -1 \quad \text{and,}$$

$$\ln(2x+2) < 0 \quad \therefore \quad 0 < 2x + 2 < 1 \quad \therefore \quad x < -\frac{1}{2}$$

3. Condition of inequality gives

$$f(x) = \frac{\ln(2x+2)}{\ln(2x^2 - x)} < 1 \quad \therefore \quad \ln(2x+2) < \ln(2x^2 - x)$$

$$\therefore \quad 2x + 2 < 2x^2 - x$$

$$\therefore \quad 0 < 2x^2 - 3x - 2$$

$$\therefore \quad 0 < 2(x+1)(x-2)$$

which is satisfied when $x > 2$, because values $x < -1$ are already excluded.

In summary, the solution set is

$$-1 < x < 0 \quad \text{and} \quad \frac{1}{2} < x < 1 \quad \text{and} \quad x > 2$$

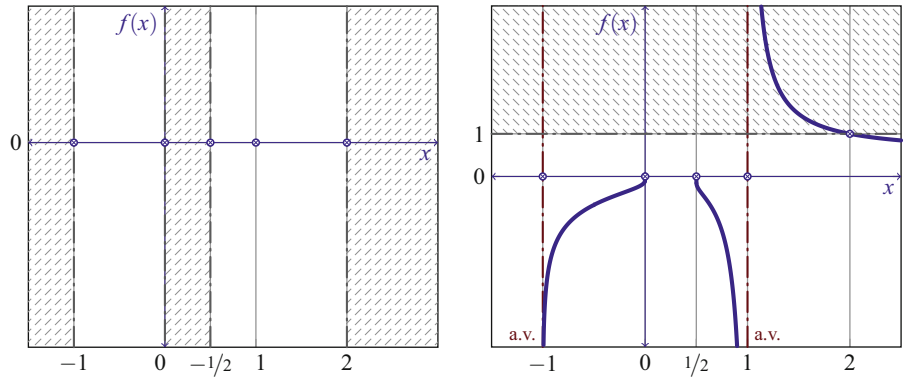


Fig. 4.14 Example P.4.64

This solution set is illustrated by non-shaded region in Fig. 4.14 (left). As a comparison, the arbitrary base logarithm term itself that is bounded by "1" is shown in Fig. 4.14 (right), where points excluded due to strict inequalities are crossed.

Trigonometry

5

All problems in this chapter are to be solved without the use of calculator. Instead, the principal objective is to master the use of "unit circle" showing "special" angles (see Fig. 5.1).

Among many trigonometric identities, some are more often used than the others, for example,

Pythagorean identities

$$a^2 + b = c^2 \quad \text{(Pythagoras's theorem)}$$

$$\sin^2\alpha + \cos^2\beta = 1 \quad \text{(Pythagoras's theorem in disguise, for } c = 1)$$

$$|\sin\alpha| = \sqrt{1 - \cos^2\alpha}$$

$$|\cos\alpha| = \sqrt{1 - \sin^2\alpha}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

Sum to product identities,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\{ \text{ if, } \alpha = \beta, \text{ (note: double angle), } \}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^{\alpha}$$

Product to sum identities,

$$\sin\alpha\sin\beta = \frac{1}{2}\Big(\cos(\alpha - \beta) - \cos(\alpha + \beta)\Big)$$

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023

R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41200-4_5

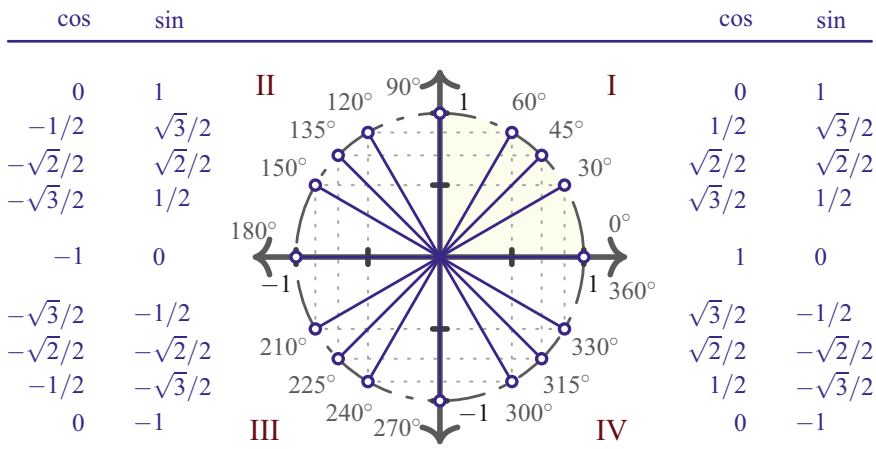


Fig. 5.1 Trigonometry, unit circle

$$\cos\alpha\cos\beta = \frac{1}{2}\Big(\cos(\alpha+\beta)+\cos(\alpha-\beta)\Big)$$
$$\sin\alpha\cos\beta = \frac{1}{2}\Big(\sin(\alpha+\beta)+\sin(\alpha-\beta)\Big)$$

Square to double angle identities,

$$\sin^2\alpha = \frac{1-\cos(2\alpha)}{2}$$
$$\cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$$

Problems

Problems in this chapter are grouped in five sections. Firstly, definitions and basic calculations with angles expressed either in degrees (°) or radians (rad) are practiced, followed by basic identities, equations, and inequalities. The chosen examples are considered classic on this topic, so they should be mastered by all engineering and science students.

5.1 Trigonometric Definitions

5.1. Briefly explain the concept of "similar triangles" (thus, "similar figures" in general), and comment their properties that are used in trigonometry.

5.2. Right-angled triangles and Pythagoras' theorem illustrate the definitions of $\sin x$, $\cos x$, and $\tan x$

5.3. Sketch the unit circle, and illustrate how angles in II, III, and IV quadrant are related to angles found in the first quadrant.

5.4. With the help of Pythagoras' theorem and unit circle, calculate $\sin x$, $\cos x$, and $\tan x$ for each of the following angles in the first quadrant: 0°, 30°, 45°, 60°, and 90°.

5.2 Basic Calculations

Convert degrees into radians and vice versa in P. 5.5 to P. 5.14.

5.5. $30°$ **5.6.** $45°$ **5.7.** $60°$ **5.8.** $90°$ **5.9.** $300°$

5.10. $330°$ **5.11.** $-18°$ **5.12.** $\frac{3\pi}{4}$ **5.13.** $\frac{5\pi}{12}$ **5.14.** $2°$

With the help of unit circle (see Fig. 5.1), calculate exact values of terms in P. 5.15 to P. 5.24.

5.15. $\tan\frac{\pi}{3}$ **5.16.** $\sin\frac{7\pi}{6}$ **5.17.** $\cos\left(-\frac{3\pi}{4}\right)$

5.18. $\cos 150°$ **5.19.** $\sin 120°$ **5.20.** $\sin\frac{\pi}{6}\sin^2\frac{\pi}{3}$

5.21. $\sin\frac{\pi}{4}\cos\frac{\pi}{4}+\cos\frac{\pi}{6}\tan\frac{\pi}{3}$ **5.22.** $\cos\frac{\pi}{4}\cos\frac{\pi}{6}+\sin\frac{\pi}{4}\cos\frac{\pi}{6}$

5.23. $\sin^2\frac{\pi}{3}-1/\tan^2\frac{\pi}{3}$ **5.24.** $\sin\left(\frac{\pi}{6}\right)\sin^2\left(\frac{\pi}{3}\right)$

In each of examples in P. 5.25 to P. 5.28, is it possible to have angle α so that both conditions are satisfied at the same time? If yes, in which quadrant?

5.25. $\sin\alpha=-\frac{1}{5}$ and $\cos\alpha=\frac{3}{5}$ **5.26.** $\sin\alpha=\frac{4}{\sqrt{17}}$ and $\cos\alpha=\frac{1}{\sqrt{17}}$

5.27. $\sin\alpha=-\frac{\sqrt{15}}{5}$ and $\cos\alpha=\frac{\sqrt{5}}{5}$ **5.28.** $\sin\alpha=-\frac{13}{14}$ and $\cos\alpha=-\frac{3\sqrt{3}}{14}$

Calculate trigonometry terms given in P. 5.29 to P. 5.40.

5.29. $\tan(-45°)$ **5.30.** $\tan(-60°)$ **5.31.** $\cos(-7\pi/6)$ **5.32.** $\sin(-315°)$

5.33. $\tan(-12\pi)$ **5.34.** $\sin(120°)$ **5.35.** $\cos(150°)$ **5.36.** $\tan(225°)$

5.37. $\tan(300°)$ **5.38.** $\tan(480°)$ **5.39.** $\tan(840°)$ **5.40.** $\cos(1320°)$

5.3 Basic Identities

Problems in this section are resolved with the help of identities. The actual list of trigonometric identities is much longer; however, the sum to product transformations, and vice versa, are among most often used. In addition, identities involving double-angle arguments are heavily used in calculus.

Given α in P. 5.41 to P. 5.46, calculate $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$.

5.41. $\alpha = 15^\circ$

5.42. $\alpha = 75^\circ$

5.43. $\alpha = 105^\circ$

5.44. $\alpha = \pi/12$

5.45. $\alpha = 5\pi/12$

5.46. $\alpha = 7\pi/12$

5.47. Calculate the exact values of $\cos\alpha$ and $\tan\alpha$ given that

$$\sin\alpha = -\frac{1}{3} \quad \text{et} \quad \pi < \alpha < \frac{3\pi}{2}$$

Calculate trigonometry products given in P. 5.48 to P. 5.57.

5.48. $\sin 20^\circ \cos 10^\circ + \cos 20^\circ \sin 10^\circ$

5.49. $\cos 43^\circ \cos 13^\circ + \sin 43^\circ \sin 13^\circ$

5.50. $-4 \sin\dfrac{5\pi}{12} \cos\dfrac{7\pi}{12}$

5.51. $\dfrac{1}{2} + 2\sin\left(1+\dfrac{\pi}{6}\right)\cos\left(1+\dfrac{\pi}{3}\right)$

5.52. $4 \sin\left(\dfrac{5\pi}{8}\right)\cos\left(\dfrac{\pi}{8}\right)$

5.53. $4\cos\left(\dfrac{5\pi}{8}\right)\cos\left(\dfrac{3\pi}{8}\right)$

5.54. $-4 \sin\left(\dfrac{7\pi}{12}\right)\sin\left(\dfrac{13\pi}{12}\right)$

5.55. $\tan 20^\circ \tan 40^\circ \tan 80^\circ$

5.56. $8 \sin 20^\circ \sin 40^\circ \sin 80^\circ$

5.57. $8 \cos 10^\circ \cos 50^\circ \cos 70^\circ$

5.58. Calculate $\sin(x+y)$ and $\sin(x-y)$ given that $\cos x = 4/5$ and $\sin y = -3/5$, where x is in IV quadrant and y is in III quadrant.

5.59. Simplify the following expression:

$$\frac{\sqrt{2}\cos\alpha - 2\cos\left(\dfrac{\pi}{4}+\alpha\right)}{2\sin\left(\dfrac{\pi}{4}+\alpha\right) - \sqrt{2}\sin\alpha}$$

Convert given trigonometric forms of products into forms of sums and given trigonometric sums into forms of products, P. 5.60 to P. 5.64

5.60. $\sin\alpha \sin\beta$ **5.61.** $\cos\alpha \cos\beta$ **5.62.** $\sin\alpha \cos\beta$

5.63. $\sin x + \sin y$ **5.64.** $\cos x + \cos y$

5.65. Given functions with double-angle arguments $\cos 2\alpha$, $\sin 2\alpha$, and $\tan 2\alpha$, derive identities that include only functions of angle argument α.

5.4 Equations

Solve equations in P. 5.66 to P. 5.73.

5.66. $\sin x = 0$ **5.67.** $\cos x = 0$ **5.68.** $\sin x = \sin\alpha$

5.69. $\sin x = \cos x$ **5.70.** $\sin x = \sin 2x$ **5.71.** $2\sin^2 x + \sin x = 0$

5.72. $\cos(x + \pi/6) = \sin(x - \pi/3)$

5.73. Solve for α in I quadrant if: $\tan\alpha = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2$.

Solve equations in P. 5.74 to P. 5.75 within given intervals.

5.74. $\cos\left(\frac{\pi}{3} - x\right) = \cos\left(\frac{\pi}{6}\right);\quad 0 < x \leq 2\pi$ **5.75.** $\sin(3x) = \sin\left(x - \frac{\pi}{2}\right);\quad -\pi < x \leq \pi$

5.5 Inequalities

Solve the following inequalities for x.

5.76. $2\cos x + 1 < 0$ **5.78.** $\cos x - \sin x < 1$ **5.80.** $|\sin x| \geq \frac{1}{2}$

5.77. $2\sin x - 1 > 0$ **5.79.** $\sin 3x - \frac{\sqrt{3}}{2} \geq 0$ **5.81.** $|\sin x| \leq \sin x + 2\cos x$

Answers

5.1 Trigonometric Definitions

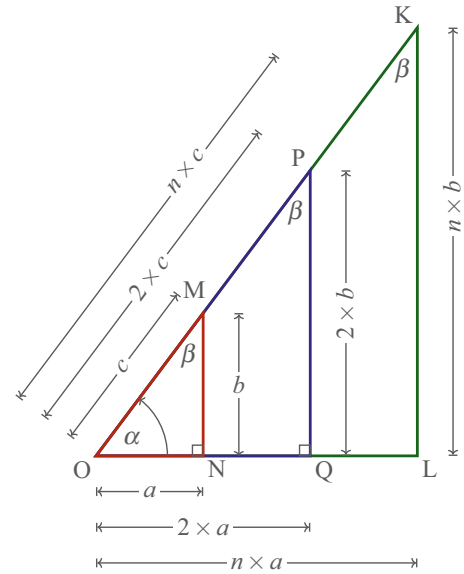
Building upon middle and high school knowledge of Pythagoras' theorem, the following examples try to emphasize its connection with fundamental trigonometric definitions and identities. Specifically, the mastery of similar right-angled triangles is essential for resolving much more complicated problems.

5.1. Two figures are said to be similar if one figure can be derived from the other by operations of uniformly scaling, translation, rotation, and/or reflection. For example, starting with triangle OMN in Fig. 5.2, triangle OPQ is derived after multiplying each side of OMN by factor of two; similarly, any other triangle can be derived by using factor n, for example, triangle OKL.

By inspection of Fig. 5.2, it is evident that, except for the scaling factors, the original triangular *form* did not change; thus, all derived figures are *similar*. By Pythagoras' theorem, it is easily verified that all three triangles are indeed right-angled, whose corresponding sides are proportional to each other. That proportionality of matching sides among similar right-angled triangles is the key to defining trigonometric functions of $\sin x$, $\cos x$, $\tan x$, ... etc.

Similarity affects angles as well. Note that α is shared by all similar triangles. Consequently, knowing two angles in a triangle, the third angle β is calculated so that their sum adds to $90° + \alpha + \beta = 180°$. In conclusion, direct consequence of similarity is that not only all figure sides are proportional, but also their corresponding angles are *equal*. The similarity of right-angled triangles is among the most important and exploited properties in mathematics.

Fig. 5.2 Example P.5.1



5.2. Evidently there are three basic proportions (ratios) related to triangle sides (see Fig. 5.2).

1. The cathetus forming α (i.e., adjacent) and hypotenuse (the side opposite the right angle) are related as

$$\frac{ON}{OM} = \frac{OQ}{OP} = \frac{OL}{OK} = \frac{a}{c} = \frac{2 \times a}{2 \times c} = \cdots = \frac{n \times a}{n \times c} = \texttt{const}.$$

2. The cathetus opposite α and hypotenuse are related as

$$\frac{MN}{OM} = \frac{PQ}{OP} = \frac{KL}{OK} = \frac{b}{c} = \frac{2 \times b}{2 \times c} = \cdots = \frac{n \times b}{n \times c} = \texttt{const}.$$

3. The cathetus opposite and adjacent to α are related as

$$\frac{MN}{ON} = \frac{PQ}{OQ} = \frac{KL}{OL} = \frac{b}{a} = \frac{2 \times b}{2 \times a} = \cdots = \frac{n \times b}{n \times a} = \texttt{const}.$$

These constant ratios of triangle sides relative to α are so important that they are named as

$$\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c} = \cos\alpha, \qquad \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c} = \sin\alpha, \qquad \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a} = \tan\alpha$$

Similarly, the equivalent ratios exist relative to β, as

$$\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c} = \cos\beta, \qquad \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} = \sin\beta, \qquad \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} = \tan\beta$$

These ratios related to α et β do not change when triangles are rotated, scaled, translated, and/or reflected. The only thing that is important is to keep track of which is "adjacent" and which is "opposite" side relative to the angle under consideration. In addition, it is useful to notice that by knowing $\sin x$ and $\cos x$, the third ratio, i.e., $\tan x$, is also known because

$$\frac{\sin x}{\cos x} = \frac{\dfrac{\text{opposite}}{\cancel{\text{hypotenuse}}}}{\dfrac{\text{adjacent}}{\cancel{\text{hypotenuse}}}} = \frac{\text{opposite}}{\text{adjacent}} = \tan x$$

Very special case, when hypotenuse $c = 1$, is extensively used as normalized size for all similar triangles. Note that, given $c = 1$, *length* of adjacent cathetus becomes *numerically equal* to the ratio named $\cos\alpha$, i.e.,

$$\frac{a}{c} = \cos\alpha \quad \therefore \quad a = c\,\cos\alpha \quad \therefore \quad (\text{if } c = 1) \quad a = \cos\alpha$$

Equally, given $c = 1$, *length* of opposite cathetus becomes *numerically equal* to the ratio named $\sin\alpha$, i.e.,

$$\frac{b}{c} = \sin\alpha \quad \therefore \quad b = c\,\sin\alpha \quad \therefore \quad (\text{if } c = 1) \quad b = \sin\alpha$$

Finally, it is useful to visualize length of adjacent cathetus as *horizontal projection* of hypotenuse and length of opposite cathetus as *vertical projection* of hypotenuse relative to α.

5.3. Trigonometric functions of sin and cos are commonly summarized in a graph known as "unit circle" (see Fig. 5.1). Note that any point at the circle is found as $x^2 + y^2 = c^2$, i.e., by Pythagoras' theorem. The circle radius, i.e., hypotenuse of the associated right-angled triangle, is normalized to one. Consequently, the length of adjacent cathetus becomes numerically equal $\cos\alpha$, and the length of opposite cathetus becomes numerically equal to $\sin\alpha$, because

$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{adjacent}}{1} \quad \therefore \quad \text{adjacent} = \cos\alpha$$
$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{opposite}}{1} \quad \therefore \quad \text{opposite} = \sin\alpha$$

Typical angles 0°, 30°, 45°, 60°, and 90° are found in the I quadrant (see Fig. 5.1). Relative to the first quadrant, typical angles in the II quadrant are shifted by 90°, i.e.,

$$30° + 90° = 120°$$
$$45° + 90° = 135°$$
$$60° + 90° = 150°$$
$$90° + 90° = 180°$$

Angles in the III quadrant are shifted again by another 90°, i.e., 180°thus

$$30° + 180° = 210°$$
$$45° + 180° = 225°$$
$$60° + 180° = 240°$$
$$90° + 180° = 270°$$

Angles in the IV quadrant are shifted again totalling 270°; thus

$$30° + 270° = 300°$$
$$45° + 270° = 315°$$
$$60° + 270° = 330°$$
$$90° + 270° = 360° = 0°$$

The symmetry among these "key" angles is clearly visible in the graph (see Fig. 5.1), where the associated right-angled triangle for each angle is clearly illustrated. Thus, the signed length (i.e., positive or negative) of horizontal and vertical catheti equate to their corresponding values of $\sin\alpha$ (horizontal) and $\cos\alpha$ (vertical).

For example, an arbitrary angle $\beta = 90° + \alpha$ in the second quadrant and its corresponding α angle in the first quadrant create similar right-angled triangles that are rotated by 90° (see Fig. 5.3). Thus, there are following identities between I and II quadrant angles (note that hypotenuse $c = 1$, and positive and negative signs of the corresponding adjunct and opposite catheti)

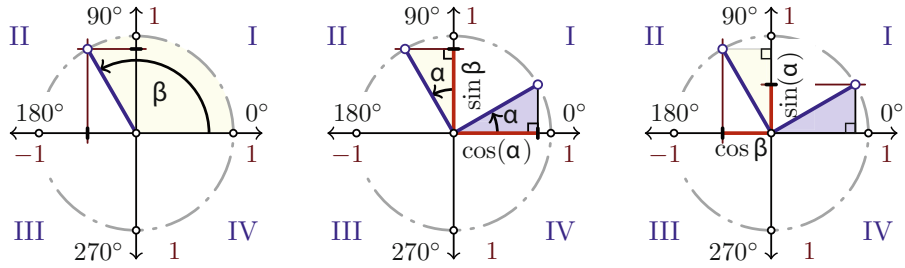
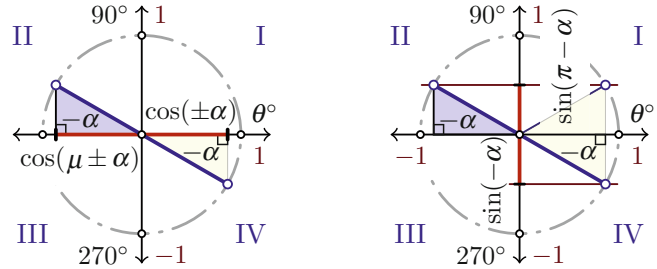


Fig. 5.3 Example P.5.3

Fig. 5.4 Example P.5.3



$$\sin\left(\alpha+\frac{\pi}{2}\right)=\sin\beta=\cos\alpha$$
$$-\cos\left(\alpha+\frac{\pi}{2}\right)=-\cos\beta=\sin\alpha$$

where the negative sign of $\cos\beta$ is direct consequence of its position in the second quadrant where the horizontal cathetus is negative. In conclusion, by knowing $\sin\alpha$ and $\cos\alpha$ in the first quadrant, it is easy to conclude that $\cos\alpha=\sin(\alpha+90°)$ and $\sin\alpha=-\cos(\alpha+90°)$. For example,

$$\left.\begin{aligned}\cos 120° &= -\sin 30°=-\frac{1}{2}\\ \sin 120° &= \cos 30°=\frac{\sqrt{3}}{2}\end{aligned}\right\}\quad\therefore\quad \tan 120°=\frac{\sqrt{3}/2}{-1/2}=-\sqrt{3}$$

Similarly, for example, angles in fourth quadrant are negative values of the equivalent angles in the first quadrant. After rotation of 180° (see Fig. 5.4) follows

$$\cos(\pi\pm\alpha)=-\cos\pm\alpha$$
$$\sin(\pi-\alpha)=-\sin(-\alpha)=\sin\alpha$$

In conclusion, knowing "key" angles in the first quadrant, unit circle is used to visually calculate sin and cos values of their corresponding angles in the other three quadrants.

5.4. Angles between 0° and 90° are said to be "in the first quadrant" of the unit circle (see Fig. 5.1). They are used to calculate trigonometric functions of angles in other three quadrants by rotating their

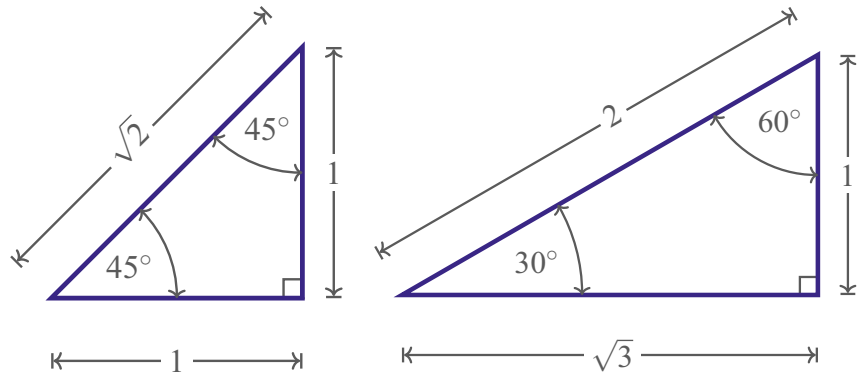


Fig. 5.5 Example P.5.4

respective right-angled triangles into the first quadrant. Specifically, right-angled triangles containing 30°, 45°, 60°, and 90° are special right triangles (see Fig. 5.5).

Case: $x = 0°$

In this special case, right-angled triangle whose hypotenuse equals one degenerates into a horizontal line segment. Consequently, the hypotenuse is found "on top" of adjacent (horizontal) cathetus; thus, the two horizontal line segments are equal. At the same time, the opposite (vertical) cathetus of the triangle is reduced to zero. Then, by definitions, it follows that:

$$\cos 0° = \frac{\cancel{\text{adjacent}}}{\cancel{\text{hypotenuse}}} = 1, \quad \sin 0° = \frac{0}{\text{hypotenuse}} = 0, \quad \tan 0° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0$$

Case: $x = 30° = \pi/6$

Special right triangle in Fig. 5.5 (left) is made so that its hypotenuse is two times longer than a one of the cathetus. Due to similarity, the two associated angles must be in the same proportion. Thus

$$\begin{aligned} 90° + \alpha + \beta &= 180° \\ \alpha &= 2\beta \quad \therefore \quad 90° + 3\beta = 180° \\ \beta &= 30° \quad \therefore \quad \alpha = 60° \end{aligned}$$

In addition, Pythagoras' theorem normalized to $c = 2$ and $a = 1$ and results in

$$2^2 = 1^2 + b^2 \quad \therefore \quad b = \sqrt{3}$$

Then, by definitions:

$$\cos 30° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}, \quad \sin 30° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2}, \quad \tan 30° = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}}$$

Case: $x = 60° = \pi/3$

The same special right triangle in Fig. 5.5 (left) is used, as

$$\cos 60° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2}, \quad \sin 60° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}, \quad \tan 60° = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1}$$

Case: $x = 45° = \pi/4$

Special right triangle in Fig. 5.5 (right) is made so that its two cathetus are equal. Thus, the two associated angles must be in the same proportion. By Pythagoras' theorem, it follows that

$$c^2 = 1^2 + 1^2 = \sqrt{2}$$

Therefore,

$$\cos 45° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}, \quad \sin 45° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}, \quad \tan 45° = \frac{\text{opposite}}{\text{adjacent}} = \frac{\cancel{1}}{\cancel{1}} = 1$$

Case: $x = 90°$

Similar to the *Case:* $x = 0°$, this special right triangle is created when its hypotenuse is found "on top" of the opposite cathetus; thus, the two vertical line segments are equal. At the same time, length of the adjacent cathetus of the triangle is reduced to zero. Thus

$$\cos 90° = \frac{0}{\text{hypotenuse}} = 0, \quad \sin 90° = \frac{\cancel{\text{opposite}}}{\cancel{\text{hypotenuse}}} = 1, \quad \tan 90° = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{0} = \infty$$

5.2 Basic Calculations

Knowing identity $180° = \pi$ rad, it follows that

5.5. $\dfrac{30°}{180°} = \dfrac{\alpha}{\pi} \;\therefore\; \alpha = \pi \dfrac{\cancel{30°}}{\cancel{180°}\,6} = \dfrac{\pi}{6}$

5.6. $\dfrac{45°}{180°} = \dfrac{\alpha}{\pi} \;\therefore\; \alpha = \pi \dfrac{\cancel{45°}}{\cancel{180°}\,4} = \dfrac{\pi}{4}$

5.7. $\dfrac{60°}{180°} = \dfrac{\alpha}{\pi} \;\therefore\; \alpha = \pi \dfrac{\cancel{60°}}{\cancel{180°}\,3} = \dfrac{\pi}{3}$

5.8. $\dfrac{45°}{180°} = \dfrac{\alpha}{\pi} \;\therefore\; \alpha = \pi \dfrac{\cancel{45°}}{\cancel{180°}\,4} = \dfrac{\pi}{4}$

5.9. $\dfrac{300°}{180°} = \dfrac{\alpha}{\pi} \;\therefore\; \alpha = \pi \dfrac{5 \times \cancel{60°}}{3 \times \cancel{60°}} = \dfrac{5\pi}{3}$

5.10. $\dfrac{330^\circ}{180^\circ} = \dfrac{\alpha}{\pi} \;\therefore\; \alpha = \pi \dfrac{3 \times 110^\circ}{3 \times 60^\circ} = \dfrac{11\pi}{6}$

5.11. $\dfrac{-18^\circ}{180^\circ} = \dfrac{\alpha}{\pi} \;\therefore\; \alpha = \pi \dfrac{-18^\circ}{10 \times 18^\circ} = -\dfrac{\pi}{10}$

5.12. $\dfrac{\alpha}{180^\circ} = \dfrac{3\pi/4}{\pi} \;\therefore\; \alpha = \dfrac{3\pi}{4\pi} 4 \times 45^\circ = 135^\circ$

5.13. $\dfrac{\alpha}{180^\circ} = \dfrac{5\pi/12}{\pi} \;\therefore\; \alpha = \dfrac{5\pi}{12\pi} 12 \times 15^\circ = 75^\circ$

5.14. $\dfrac{2^\circ}{180^\circ} = \dfrac{\alpha}{\pi} \;\therefore\; \alpha = \pi \dfrac{2^\circ\ 1}{90 \times 2^\circ} = \dfrac{\pi}{90}$

5.15. By inspection of unit circle in Fig. 5.1, it is straightforward to find that angle $\pi/3$ is in the first quadrant; thus

$$\tan\frac{\pi}{3} = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

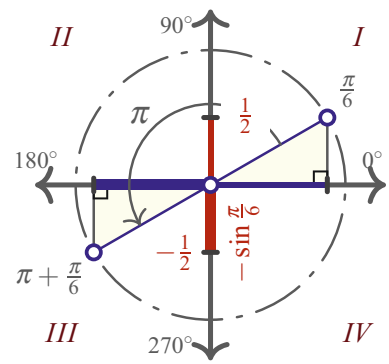
5.16. By inspection of unit circle in Fig. 5.1, it is straightforward to find angle of $7\pi/6$ in the third quadrant, so that

$$\sin\frac{7\pi}{6} = \sin\left(\frac{6\pi + \pi}{6}\right) = \sin\left(\pi + \frac{\pi}{6}\right) \equiv 210^\circ$$

As evident by inspection of unit circle, in the third quadrant, $\pi + \pi/6$ angle has both vertical (sin) and horizontal (cos) projections negative (see Fig. 5.6); thus

$$\sin\frac{7\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

Fig. 5.6 Example P.5.16



5.17. $\cos\left(-\frac{3\pi}{4}\right) = \cos\left(\frac{\pi}{4} \pm \pi\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

5.18. $\cos 150^\circ = \cos(180^\circ - 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$

5.19. $\sin 120^\circ = \cos(90^\circ + 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$

5.20. $\sin\frac{\pi}{6}\sin^2\frac{\pi}{3} = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{2}\frac{3}{4} = \frac{3}{8}$

5.21. $\sin\frac{\pi}{4}\cos\frac{\pi}{4} + \cos\frac{\pi}{6}\tan\frac{\pi}{3} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\sqrt{3} = \frac{1}{2} + \frac{3}{2} = 2$

5.22. $\cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\cos\frac{\pi}{6} = \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} = \sqrt{\frac{3}{2}}$

5.23. $\sin^2\frac{\pi}{3} - \frac{1}{\tan^2\frac{\pi}{3}} = \left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{\left(\sqrt{3}\right)^2} = \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}$

5.24. $\sin\left(\frac{\pi}{6}\right)\sin^2\left(\frac{\pi}{3}\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{2}\frac{3}{4} = \frac{3}{8}$

5.25. Assuming a, b to be the two catheti and c hypotenuse and knowing that hypotenuse of a right-angled triangle equals the unit circle radius, i.e., $c = 1$, then by Pythagoras' theorem,

$$a = \cos\alpha, \quad b = \sin\alpha \quad \therefore \quad a^2 + b^2 = 1^2 \quad \therefore \quad \cos^2\alpha + \sin^2\alpha = 1^2$$

which is the trigonometric form of Pythagoras' theorem. Therefore, a simple verification results in

$$\sin^2\alpha + \cos^2\alpha = \left(-\frac{1}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{1}{25} + \frac{9}{25} = \frac{\cancel{10}\,2}{\cancel{25}\,5} = \frac{2}{5} \neq 1 \quad \times$$

that is to say, it is not possible to have angle α whose $\sin\alpha = -1/5$ and $\cos\alpha = 3/5$.

5.26. $\sin^2\alpha + \cos^2\alpha = \left(\frac{4}{\sqrt{17}}\right)^2 + \left(\frac{1}{\sqrt{17}}\right)^2 = \frac{16}{17} + \frac{1}{17} = \frac{17}{17} = 1 \quad \checkmark$

Both vertical ($\sin\alpha$) and horizontal ($\cos\alpha$) projections are positive; therefore, α is in the I quadrant.

5.27. $\sin^2\alpha + \cos^2\alpha = \left(-\frac{\sqrt{15}}{5}\right)^2 + \left(\frac{\sqrt{5}}{5}\right)^2 = \frac{15}{25} + \frac{5}{25} = \frac{\cancel{20}\,4}{\cancel{25}\,5} = \frac{4}{5} \neq 1 \quad \times$

that is to say, it is not possible to have angle α whose $\sin\alpha = -\sqrt{15}/5$ and $\cos\alpha = \sqrt{5}/5$.

5.28. $\sin^2\alpha + \cos^2\alpha = \left(-\frac{13}{14}\right)^2 + \left(-\frac{3\sqrt{3}}{14}\right)^2 = \frac{169}{196} + \frac{27}{196} = \frac{196}{196} = 1$ ✓

Both vertical ($\sin\alpha$) and horizontal ($\cos\alpha$) projections are negative; therefore, α is in the *III* quadrant.

5.29. It is often possible to decompose given angles into linear combination of special angles given in the unity circle, whose sin and cos (therefore, also tan) values are already known. In addition, it is very important to visually understand relations among similar angles in different quadrants.

For example, note that horizontal projections, i.e., $\cos(\pm\alpha)$, are the same length, as the consequence of similarity of their respective right-angled triangles. Thus it is true that

$$\cos\alpha = \cos(-\alpha)$$

Similarly, angles of $\pi \pm \alpha$ and $\pm\alpha$ are related as in Fig. 5.7. It is evident that the horizontal projections (i.e., cos) are the same length; however, they have *opposite signs* (see Fig. 5.7 (left)). Similarly, vertical projections (i.e., sin) also have *opposite signs* (see Fig. 5.7 (right)); thus, it's true that

$$\cos(\pm\alpha) = -\cos(\pi \pm \alpha)$$
$$\sin(\pm\alpha) = -\sin(\pi \pm \alpha)$$

Or, for example, relations between angles of α and $90° + \alpha$ are also important (see Fig. 5.8). It is evident that the vertical projection, i.e., $\sin\alpha$ (in the first quadrant and positive) is the same length as the horizontal projection, i.e., $\cos(90° + \alpha)$ (in the second quadrant and negative) (see Fig. 5.8 (left)), as their respective similar right-angled triangles are rotated by 90°. Or identity between $\cos\alpha$ and $\sin(90° + \alpha)$, both positive, is found as in Fig. 5.8 (right); thus, it is true that

$$\sin(\alpha) = -\sin(90° + \alpha)$$
$$\cos(\alpha) = \sin(90° + \alpha)$$

Reminder: In summary, there are many trigonometric identities that are possible; thus, memorizing them all requires extraordinary effort. Instead, after mastering the use of unit circle, all these identities become evident.

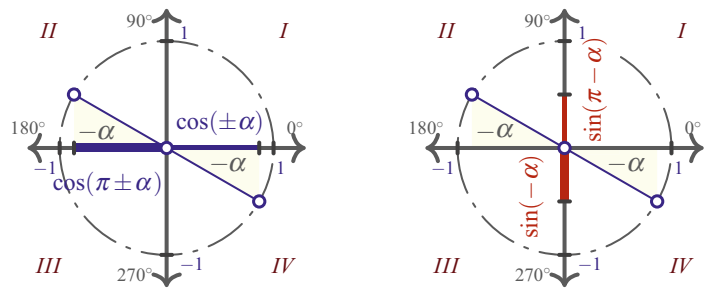
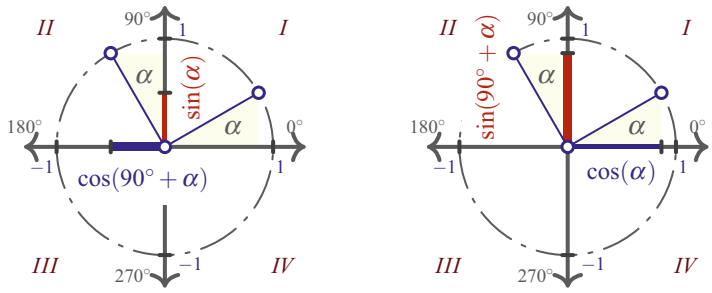


Fig. 5.7 Example P.5.29

Fig. 5.8 Example P.5.29



In this example P. 5.29, by inspection of unit circle in Fig. 5.1, sin and cos of negative angle are

$$\tan(-45°) = \frac{\sin(-45°)}{\cos(-45°)} = \frac{-\sin(45°)}{\cos(45°)} = -\frac{\sin(45°)}{\cos(45°)} = -\frac{\sqrt{2}/2}{\sqrt{2}/2} = -1$$

5.30. By inspection of unit circle in Fig. 5.1, sin and cos of negative angle are

$$\tan(-60°) = \frac{\sin(-60°)}{\cos(-60°)} = \frac{-\sin(60°)}{\cos(60°)} = -\frac{\sin(60°)}{\cos(60°)} = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3}$$

5.31. Given $-7\pi/6$ angle is equivalent to $5\pi/6$ in the positive direction, or 150°, either way, by inspection of unity circle, it follows that

$$\cos\left(-\frac{7\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = \cos(150°) = -\frac{\sqrt{3}}{2}$$

5.32. Given $-315°$ angle is equivalent to 45° from the positive side, thus

$$\sin(-315°) = \sin(45°) = \frac{\sqrt{2}}{2}$$

5.33. Given -12π angle is equivalent to $(-6 \times 2\pi) = 0$, thus

$$\tan(-12\pi) = \tan(0) = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

5.34. Given 120° angle may be decomposed as sum of 90° and 30° (see Fig. 5.8), as

$$\sin(120°) = \sin(90° + 30°) = \{\cos(\alpha) = \sin(90° + \alpha)\} = \cos(30°) = \frac{\sqrt{3}}{2}$$

5.35. Angles outside of the first quadrant range may be decomposed into linear combination of special angles. Then, it is possible to use trigonometric identities and resolve the problem. Here, 150° angle may be decomposed as difference of 180° and 30° (see Fig. 5.7) as

$$\cos(150°) = \cos(180° - 30°) = \{\cos(\pi \pm \alpha) = -\cos(\alpha)\} = -\cos(30°) = -\frac{\sqrt{3}}{2}$$

5.36. Given 225° angle may be decomposed as sum of 180° and 45° (see Fig. 5.7) as

$$\tan(225^\circ) = \tan(180^\circ + 45^\circ) = \frac{\sin(180^\circ + 45^\circ)}{\cos(180^\circ + 45^\circ)} = \frac{-\sin(45^\circ)}{-\cos(45^\circ)} = \frac{\sin(45^\circ)}{\cos(45^\circ)} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

5.37. Given 300° angle may be decomposed as difference of 360° and 60°. Note that angles are periodic: that is to say that after each full rotation, either in positive or negative direction, the rotating point ends up at the original position α.

$$\begin{aligned}\tan(300^\circ) &= \tan(360^\circ - 60^\circ) = \left\{\pm n \times 2\pi + \alpha = \alpha\right\} = \tan(-60^\circ) = \frac{\sin(-60^\circ)}{\cos(-60^\circ)} \\ &= \left\{\cos(-\alpha) = \cos(\alpha),\quad \sin(-\alpha) = -\sin(\alpha)\right\} \\ &= \frac{-\sin(60^\circ)}{\cos(60^\circ)} = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3}\end{aligned}$$

5.38. Given 480° angle may be decomposed as combination of 360°, 120°, and 60°,

$$\begin{aligned}\tan(480^\circ) &= \tan(360^\circ + 120^\circ) = \tan(180^\circ - 60^\circ) = \tan(-60^\circ) = \frac{\sin(-60^\circ)}{\cos(-60^\circ)} \\ &= \frac{-\sin(60^\circ)}{\cos(60^\circ)} = -\frac{\sqrt{2}/2}{1/2} = -\sqrt{3}\end{aligned}$$

5.39. Given 840° angle may be decomposed, for example, as combination of 360°, 180°, 120°, and 60°,

$$\begin{aligned}\tan(840^\circ) &= \tan(2 \times 360^\circ + 120^\circ) = \tan(120^\circ) = \tan(180^\circ - 60^\circ) = \tan(-60^\circ) \\ &= -\tan(60^\circ) = -\sqrt{3}\end{aligned}$$

5.40. Given 1320° angle may be decomposed, for example, as combination of 360°, 240°, 180°, and 60°,

$$\cos(1320^\circ) = \cos(3 \times 360^\circ + 240^\circ) = \cos(180^\circ + 60^\circ) = -\cos(60^\circ) = -\frac{1}{2}$$

5.3 Basic Identities

5.41. Functions of sum or difference arguments may be resolved with the help of trigonometric identities. For example, angle of 15° can be decomposed as (45° − 30°), so that

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ) \\ &\left\{\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta\right\}\end{aligned}$$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$$

$$\begin{aligned}
\cos 15^\circ &= \cos(45^\circ - 30^\circ) \\
&\quad \{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\} \\
&= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\
&= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}}{4}(\sqrt{3}+1) \\
\tan 15^\circ &= \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\cancel{\frac{\sqrt{2}}{4}}(\sqrt{3}-1)}{\cancel{\frac{\sqrt{2}}{4}}(\sqrt{3}+1)} \quad \frac{\sqrt{3}-1}{\sqrt{3}-1} \\
&= \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{3-2\sqrt{3}+1}{3-1} = \frac{\cancel{2}(2-\sqrt{3})}{\cancel{2}} = 2-\sqrt{3}
\end{aligned}$$

5.42. Angle of 75° may be decomposed as $(45^\circ + 30^\circ)$, so that

$$\begin{aligned}
\sin 75^\circ &= \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\
&= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}}{4}(\sqrt{3}+1) \\
\cos 75^\circ &= \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\
&= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}}{4}(\sqrt{3}-1) \\
\tan 75^\circ &= \frac{\sin 75^\circ}{\cos 75^\circ} = \frac{\cancel{\frac{\sqrt{2}}{4}}(\sqrt{3}+1)}{\cancel{\frac{\sqrt{2}}{4}}(\sqrt{3}-1)} \quad \frac{\sqrt{3}+1}{\sqrt{3}+1} \\
&= \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{3+2\sqrt{3}+1}{3-1} = \frac{\cancel{2}(2+\sqrt{3})}{\cancel{2}} = 2+\sqrt{3}
\end{aligned}$$

5.43. Angle of 105° may be seen as $(60^\circ + 45^\circ)$, so that

$$\begin{aligned}
\sin 105^\circ &= \sin(60^\circ + 45^\circ) = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ \\
&= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}(\sqrt{3}+1) \\
\cos 105^\circ &= \cos(60^\circ + 45^\circ) = \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ \\
&= \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}(1-\sqrt{3})
\end{aligned}$$

$$\tan 105^\circ = \frac{\sin 105^\circ}{\cos 105^\circ} = \frac{\frac{\sqrt{2}}{4}(\sqrt{3}+1)}{\frac{\sqrt{2}}{4}(1-\sqrt{3})} \quad \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{3+2\sqrt{3}+1}{1-3} = \frac{2(2+\sqrt{3})}{-2} = -(2+\sqrt{3})$$

5.44. Angle of $\pi/12$ may be seen as $(\pi/3 - \pi/4)$, so that

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$$

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$$

$$\tan\frac{\pi}{12} = \frac{\sin\frac{\pi}{12}}{\cos\frac{\pi}{12}} = \frac{\frac{\sqrt{2}}{4}(\sqrt{3}-1)}{\frac{\sqrt{2}}{4}(\sqrt{3}+1)} \quad \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = 2-\sqrt{3}$$

5.45. Angle of $5\pi/12$ may be seen as $(\pi/4 + \pi/6)$, so that

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \cdots = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$$

$$\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \cdots = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$$

$$\tan\left(\frac{5\pi}{12}\right) = \frac{\sin\left(\frac{5\pi}{12}\right)}{\cos\left(\frac{5\pi}{12}\right)} = \frac{\frac{\sqrt{2}}{4}(\sqrt{3}+1)}{\frac{\sqrt{2}}{4}(\sqrt{3}-1)} = \cdots = 2+\sqrt{3}$$

5.46. Angle of $7\pi/12$ may be seen as $(\pi/3 + \pi/4)$, so that

$$\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \cdots = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$$

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \cdots = \frac{\sqrt{2}}{4}(1-\sqrt{3})$$

$$\tan\left(\frac{7\pi}{12}\right) = \frac{\sin\left(\frac{7\pi}{12}\right)}{\cos\left(\frac{7\pi}{12}\right)} = \cdots = \frac{\sqrt{3}+1}{1-\sqrt{3}} = -(2+\sqrt{3})$$

5.47. Knowing the trigonometric form of Pythagoras' theorem

$$\cos^2\alpha + \sin^2\alpha = 1^2$$

and given

$$\sin\alpha = -\frac{1}{3} \quad \text{et} \quad \pi < \alpha < \frac{3\pi}{2}$$

it follows that

$$\cos^2\alpha + \sin^2\alpha = 1 \quad \therefore \quad \cos^2\alpha = 1 - \sin^2\alpha = 1 - \left(-\frac{1}{3}\right)^2$$

$$\cos\alpha = \sqrt{1-\frac{1}{9}} \quad \therefore \quad \cos\alpha = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$

However, α is in the *third* quadrant, i.e., $\cos x$ is *negative* (see Fig. 5.1, i.e. $\cos\alpha = -2\sqrt{2}/3$).

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-1/\cancel{3}}{-2\sqrt{2}/\cancel{3}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

5.48. Inter-products of $\sin(x)$ and $\cos(x)$ may be resolved with trigonometric identities, as

$$\sin 20^\circ \cos 10^\circ + \cos 20^\circ \sin 10^\circ =$$

$$\left\{\sin\alpha\cos\beta = \frac{1}{2}\big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\big)\right\}$$

$$= \frac{1}{2}\sin(20^\circ + 10^\circ) + \frac{1}{2}\sin(20^\circ - 10^\circ) + \frac{1}{2}\sin(10^\circ + 20^\circ) + \frac{1}{2}\sin(10^\circ - 20^\circ)$$

$$= \frac{1}{2}\sin(30^\circ) + \frac{1}{2}\cancel{\sin(10^\circ)} + \frac{1}{2}\sin(30^\circ) + \frac{1}{2}\cancel{\sin(-10^\circ)} = \big\{\sin(-x) = -\sin(x)\big\}$$

$$= \sin 30^\circ = \frac{1}{2}$$

5.49. Inter-products of $\sin(x)$ and $\cos(x)$ may be resolved with trigonometric identities, as

$$\cos 43^\circ \cos 13^\circ + \sin 43^\circ \sin 13^\circ =$$

$$\left\{\begin{array}{l} \cos\alpha\cos\beta = \frac{1}{2}\big(\cos(\alpha+\beta) + \cos(\alpha-\beta)\big) \\ \sin\alpha\sin\beta = \frac{1}{2}\big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\big) \end{array}\right\}$$

$$= \frac{1}{2}\cos(43^\circ + 13^\circ) + \frac{1}{2}\cos(43^\circ - 13^\circ) + \frac{1}{2}\cos(43^\circ - 13^\circ) - \frac{1}{2}\cos(43^\circ + 13^\circ)$$

$$= \frac{1}{2}\cancel{\cos(56^\circ)} + \frac{1}{2}\cos(30^\circ) + \frac{1}{2}\cos(30^\circ) - \frac{1}{2}\cancel{\cos(56^\circ)} = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

5.50. Inter-products of $\sin(x)$ and $\cos(x)$ may be resolved with trigonometric identities, as

$$-4\sin\frac{5\pi}{12}\cos\frac{7\pi}{12} = \left\{\sin\alpha\cos\beta = \frac{1}{2}\big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\big)\right\}$$

$$= -\cancel{4}\,\overset{2}{}\,\frac{1}{\cancel{2}}\left[\sin\left(\frac{5\pi}{12} + \frac{7\pi}{12}\right) + \sin\left(\frac{5\pi}{12} - \frac{7\pi}{12}\right)\right]$$

$$= -2\left[\sin\left(\frac{\cancel{12}\pi}{\cancel{12}}\right) + \sin\left(-\frac{\cancel{2}\pi}{\cancel{12}\,6}\right)\right] \left\{\sin(-x) = -\sin(x)\right\}$$

$$= -2\left[0 - \sin\left(\frac{\pi}{6}\right)\right] = -2\left(-\frac{1}{2}\right) = 1$$

5.51. Inter-products of $\sin(x)$ and $\cos(x)$ may be resolved with trigonometric identities, as

$$\frac{1}{2} + 2\sin\left(1 + \frac{\pi}{6}\right)\cos\left(1 + \frac{\pi}{3}\right) = \left\{\sin\alpha\cos\beta = \frac{1}{2}\big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\big)\right\}$$

$$= \frac{1}{2} + \left[\sin\left(1 + \frac{\pi}{6} + 1 + \frac{\pi}{3}\right) + \sin\left(\cancel{1} + \frac{\pi}{6} - \cancel{1} - \frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2} + \left[\sin\left(2 + \frac{\pi}{2}\right) + \sin\left(-\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2} + \left[\sin\left(\frac{\pi}{2} + 2\right) - \sin\left(\frac{\pi}{6}\right)\right]$$

$$\left\{\sin\left(\frac{\pi}{2} + x\right) = \cos(x), \quad \text{(see unit circle and Fig. 5.3)}\right\}$$

$$= \frac{1}{2} + \left[\cos(2) - \frac{1}{2}\right]$$

$$= \cos(2)$$

5.52. Inter-products of $\sin(x)$ and $\cos(x)$ may be resolved with trigonometric identities, as

$$4\sin\left(\frac{5\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) = \left\{\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)\right\}$$

$$= \cancel{4}\,\frac{1}{\cancel{2}}\left[\sin\left(\frac{5\pi}{8} + \frac{\pi}{8}\right) + \sin\left(\frac{5\pi}{8} - \frac{\pi}{8}\right)\right]$$

$$= 2\left[\sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right)\right] = \left\{\text{(see unit circle)}\right\} = 2\left[\frac{\sqrt{2}}{2} + 1\right]$$

$$= \sqrt{2} + 2$$

5.53. Inter-products of $\sin(x)$ and $\cos(x)$ may be resolved with trigonometric identities, as

$$\begin{aligned}
4\cos\left(\frac{5\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right) &= \left\{\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right)\right\} \\
&= \cancel{4}\,\frac{1}{\cancel{2}}\left[\cos\left(\frac{5\pi}{8}+\frac{3\pi}{8}\right)+\cos\left(\frac{5\pi}{8}-\frac{3\pi}{8}\right)\right] \\
&= 2\left[\cos(\pi)+\cos\left(\frac{\pi}{4}\right)\right] = \left\{\text{(see unit circle)}\right\} = 2\left[-1+\frac{\sqrt{2}}{2}\right] \\
&= \sqrt{2}-2
\end{aligned}$$

5.54. Inter-products of $\sin(x)$ and $\cos(x)$ may be resolved with trigonometric identities, as

$$\begin{aligned}
-4\sin\left(\frac{7\pi}{12}\right)\sin\left(\frac{13\pi}{12}\right) &= \left\{\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)\right\} \\
&= -\cancel{4}\,\cancel{2}\,\frac{1}{\cancel{2}}\left[\cos\left(\frac{7\pi}{12}-\frac{13\pi}{12}\right)-\cos\left(\frac{7\pi}{12}+\frac{13\pi}{12}\right)\right] \\
&= -2\left[\cos\left(-\frac{\cancel{6}\pi}{\cancel{12}\,\cancel{2}}\right)-\cos\left(\frac{\cancel{20}\,5\pi}{\cancel{12}\,3}\right)\right] \\
&= -2\left[\cos\left(\frac{\pi}{2}\right)^{0}-\cos\left(\frac{5\pi}{3}\right)\right] \\
&= -2\left[-\cos\left(-\frac{\pi}{3}\right)\right] = -2\left(-\frac{1}{2}\right) = 1
\end{aligned}$$

5.55. After replacing $\tan(x)$ functions with their equivalent $\sin(x)/\cos(x)$ ratios, the product identities can be used to resolve numerator and denominator separately, as

$$\tan 20^\circ\ \tan 40^\circ\ \tan 80^\circ = \frac{\sin 20^\circ}{\cos 20^\circ}\frac{\sin 40^\circ}{\cos 40^\circ}\frac{\sin 80^\circ}{\cos 80^\circ}$$

Then, numerator products are,

$$\begin{aligned}
\sin 20^\circ \sin 40^\circ \sin 80^\circ &= \left\{\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)\right\} \\
&= \frac{1}{2}\big(\cos(20^\circ-40^\circ)-\cos(20^\circ+40^\circ)\big)\sin 80^\circ \\
&= \frac{1}{2}\left[\cos(-20^\circ)^{\cos 20^\circ}-\cos(60^\circ)\right]\sin 80^\circ \\
&= \frac{1}{2}\left[\cos 20^\circ\ \sin 80^\circ-\cos(60^\circ)\ \sin 80^\circ\right]\quad\text{(cont.)}
\end{aligned}$$

$$\left\{\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)\right\}$$

$$= \frac{1}{2}\left[\frac{1}{2}\Big(\sin 100^\circ + \cancel{\sin 60^\circ}^{\sqrt{3}/2}\Big) - \cancel{\cos 60^\circ}^{1/2}\ \sin 80^\circ\right]$$

$$= \frac{1}{4}\left[\sin(90^\circ+10^\circ) + \frac{\sqrt{3}}{2} - \sin(90^\circ-10^\circ)\right]$$

$$\left\{\sin(90^\circ+\alpha) = \sin(90^\circ-\alpha),\quad \text{(see unit circle)}\right\}$$

$$= \frac{1}{4}\left[\cancel{\sin(90^\circ+10^\circ)} + \frac{\sqrt{3}}{2} - \cancel{\sin(90^\circ-10^\circ)}\right]$$

$$= \frac{\sqrt{3}}{8}$$

Similarly, note that $\cos(90^\circ+10^\circ) = -\cos(90^\circ-10^\circ)$, as per unit circle, so that

$$\begin{aligned}
\cos 20^\circ\cos 40^\circ\cos 80^\circ &= \frac{1}{2}\Big[\cos(20^\circ+40^\circ)+\cos(20^\circ-40^\circ)\Big]\cos 80^\circ\\
&= \frac{1}{2}\Big[\cancel{\cos 60^\circ}^{1/2} + \cancel{\cos(-20^\circ)}^{\cos 20^\circ}\Big]\cos 80^\circ\\
&= \frac{1}{2}\Big[\frac{1}{2}\cos 80^\circ + \cos 20^\circ\cos 80^\circ\Big]\\
&= \frac{1}{2}\Big[\underbrace{\frac{1}{2}\cancel{\cos 80^\circ}}_{\cos(90-10)^\circ} + \underbrace{\frac{1}{2}\cancel{\cos 100^\circ}}_{\cos(90+10)^\circ} + \frac{1}{2}\cancel{\cos 60^\circ}^{1/2}\Big]\\
&= \frac{1}{8}
\end{aligned}$$

Therefore,

$$\tan 20^\circ\ \tan 40^\circ\ \tan 80^\circ = \frac{\sqrt{3}/\cancel{8}}{1/\cancel{8}} = \sqrt{3}$$

5.56. In a similar manner as in P. 5.55,

$$\begin{aligned}
8\sin 20^\circ\sin 40^\circ\sin 80^\circ &= 4\big[\cos(20^\circ-40^\circ)-\cos(20^\circ+40^\circ)\big]\sin 80^\circ\\
&= 4\big[\sin 80^\circ\cos(-20^\circ)-\sin 80^\circ\cos 60^\circ\big]\\
&= 4\left[\sin 80^\circ\cos(20^\circ) - \frac{1}{2}\sin 80^\circ\right]\\
&= 4\left[\frac{1}{2}\sin 100^\circ + \frac{1}{2}\sin 60^\circ - \frac{1}{2}\sin 80^\circ\right]\\
&\left\{\sin 100^\circ = \sin(90^\circ+10^\circ) = \sin(90^\circ-10^\circ) = \sin 80^\circ\right\}\\
&= \cancel{4}\left[\frac{\cancel{\sin(90^\circ+10^\circ)}}{2} + \frac{\sqrt{3}}{\cancel{4}} - \frac{\cancel{\sin(90^\circ-10^\circ)}}{2}\right] = \sqrt{3}
\end{aligned}$$

5.57. In a similar manner as in P. 5.55,

$$\begin{aligned}
8\cos 10^\circ \cos 50^\circ \cos 70^\circ &= 4\big[\cos 60^\circ + \cos 40^\circ\big]\cos 70^\circ \\
&= 4\left[\frac{1}{2}\cos 70^\circ + \cos 40^\circ \cos 70^\circ\right] \\
&= 4\left[\frac{1}{2}\cos 70^\circ + \frac{1}{2}\cos 110^\circ + \frac{1}{2}\cos 30^\circ\right] \\
&= 4\left[\frac{1}{2}\cos 70^\circ + \frac{1}{2}\cos 110^\circ + \frac{1}{2}\frac{\sqrt{3}}{2}\right] \\
&\big\{\cos 110^\circ = \cos(90^\circ + 20^\circ) = -\cos(90^\circ - 20^\circ) = -\cos 70^\circ\big\} \\
&= 2\left[\cancel{\cos 70^\circ} - \cancel{\cos 70^\circ} + \frac{\sqrt{3}}{\cancel{2}}\right] \\
&= \sqrt{3}
\end{aligned}$$

5.58. Given Pythagoras' theorem, unit circle in Fig. 5.1, and the sum/difference identities, then

$$\left\{\begin{array}{l}\sin^2 x + \cos^2 x = 1 \\ \sin(x+y) = \sin x \cos y + \cos x \sin y \\ \sin(x-y) = \sin x \cos y - \cos x \sin y\end{array}\right\}$$

$$\left.\begin{array}{lll}\cos x &= 4/5 \;\;\therefore\;\; \sin x = \sqrt{1-(4/5)^2} = 3/5 \\ \sin y &= -3/5 \;\;\therefore\;\; \cos y = \sqrt{1-(-3/5)^2} = 4/5\end{array}\right\} \;\therefore\; \left\{\begin{array}{lll}\sin x &= -\dfrac{3}{5} & \text{(IV)} \\ \cos y &= -\dfrac{4}{5} & \text{(III)}\end{array}\right.$$

Therefore,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y = \left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right) + \left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = \frac{12}{25} - \frac{12}{25} = 0$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y = \left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right) - \left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = \frac{12}{25} + \frac{12}{25} = \frac{24}{25}$$

5.59. Given Pythagoras' theorem, unit circle in Fig. 5.1, and the sum/difference identities, it follows that

$$\frac{\sqrt{2}\cos\alpha - 2\cos\left(\frac{\pi}{4}+\alpha\right)}{2\sin\left(\frac{\pi}{4}+\alpha\right) - \sqrt{2}\sin\alpha} = \frac{2\left(\frac{\sqrt{2}}{2}\cos\alpha - \cos\left(\frac{\pi}{4}+\alpha\right)\right)}{2\left(\sin\left(\frac{\pi}{4}+\alpha\right) - \frac{\sqrt{2}}{2}\sin\alpha\right)}$$

$$\left\{\begin{array}{l}\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{array}\right\}$$

(cont.)

$$= \frac{2\left(\cos\frac{\pi}{4}\cos\alpha - \cos\frac{\pi}{4}\cos\alpha + \sin\frac{\pi}{4}\sin\alpha\right)}{2\left(\sin\frac{\pi}{4}\cos\alpha + \cos\frac{\pi}{4}\sin\alpha - \cos\frac{\pi}{4}\sin\alpha\right)}$$

$$= \frac{\sin\frac{\pi}{4}\sin\alpha}{\sin\frac{\pi}{4}\cos\alpha} = \tan\alpha$$

5.60. Given the sum/difference identities,

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

it follows that

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta - \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$= 2\sin\alpha\sin\beta$$
$$\therefore$$
$$\sin\alpha\sin\beta = \frac{1}{2}\big(\cos(\alpha - \beta) - \cos(\alpha + \beta)\big)$$

5.61. Given the sum/difference identities,

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

it follows that

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$= 2\cos\alpha\cos\beta$$
$$\therefore$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha + \beta) + \cos(\alpha - \beta)\right)$$

5.62. Given the sum/difference identities,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

it follows that

$$\begin{aligned}\sin(\alpha+\beta)+\sin(\alpha-\beta) &= \sin\alpha\cos\beta+\cancel{\cos\alpha\sin\beta}+\sin\alpha\cos\beta-\cancel{\cos\alpha\sin\beta}\\ &= 2\sin\alpha\cos\beta\\ &\therefore\\ \sin\alpha\cos\beta &= \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)\end{aligned}$$

Note that identities in P. 5.60 to P. 5.62 are fundamental for RF electronics and communication theory.

5.63. Given the sum identities,

$$\sin\alpha\cos\beta=\frac{1}{2}\big(\sin(\alpha+\beta)+\sin(\alpha-\beta)\big)\ \ \therefore\ \ \sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$$

$$\left\{\begin{aligned}\alpha+\beta &= x\\ \alpha-\beta &= y\end{aligned}\right\}\ \therefore\ \left\{\begin{aligned}\alpha &= \frac{x+y}{2}\\ \beta &= \frac{x-y}{2}\end{aligned}\right\}$$

$$\therefore$$

$$\sin x+\sin y=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Or, in general

$$\sin\alpha+\sin\beta=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

5.64. Given the sum identities

$$\cos\alpha\cos\beta=\frac{1}{2}\big(\cos(\alpha+\beta)+\cos(\alpha-\beta)\big)\ \ \therefore\ \ \cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta$$

$$\left\{\begin{aligned}\alpha+\beta &= x\\ \alpha-\beta &= y\end{aligned}\right\}\ \therefore\ \left\{\begin{aligned}\alpha &= \frac{x+y}{2}\\ \beta &= \frac{x-y}{2}\end{aligned}\right\}$$

$$\therefore$$

$$\cos x+\cos y=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Or, in general

$$\cos\alpha+\cos\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

5.65. Given trigonometric identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

end setting $\alpha = \beta$ it follows that

$$\sin(\alpha+\alpha) = \underline{\sin 2\alpha} = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha = \underline{2\sin\alpha\cos\alpha}$$
$$\cos(\alpha+\alpha) = \underline{\cos 2\alpha} = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha = \underline{\cos^2\alpha - \sin^2\alpha}$$

$$\therefore$$

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{\sin\alpha\cos\alpha + \cos\alpha\sin\alpha}{\cos\alpha\cos\alpha - \sin\alpha\sin\alpha} = \frac{\sin\alpha\cos\alpha + \cos\alpha\sin\alpha}{\cos\alpha\cos\alpha\left(1 - \frac{\sin\alpha}{\cos\alpha}\frac{\sin\alpha}{\cos\alpha}\right)}$$

$$= \frac{\frac{\sin\alpha\,\cancel{\cos\alpha}}{\cos\alpha\,\cancel{\cos\alpha}} + \frac{\cancel{\cos\alpha}\,\sin\alpha}{\cancel{\cos\alpha}\,\cos\alpha}}{1 - \tan^2\alpha}$$

$$= \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

Furthermore,

$$\left.\begin{aligned}\cos^2\alpha + \sin^2\alpha &= 1\\ \cos^2\alpha - \sin^2\alpha &= \cos 2\alpha\end{aligned}\right\} \therefore \begin{cases} 2\sin^2\alpha = 1 - \cos 2\alpha & \therefore \ \sin^2\alpha = \dfrac{1-\cos 2\alpha}{2} \\ 2\cos^2\alpha = 1 + \cos 2\alpha & \therefore \ \cos^2\alpha = \dfrac{1+\cos 2\alpha}{2} \end{cases}$$

In addition,

$$\sin 2\alpha = 2\sin\alpha\cos\alpha\ \frac{\cos\alpha}{\cos\alpha} = 2\frac{\sin\alpha}{\cos\alpha}\ \frac{\cos^2\alpha}{1} = 2\tan\alpha\frac{\cos^2\alpha}{\cos^2\alpha + \sin^2\alpha} = 2\tan\alpha\ \frac{1}{\frac{\cos^2\alpha}{\cos^2\alpha} + \frac{\sin^2\alpha}{\cos^2\alpha}}$$

$$= \frac{2\tan\alpha}{1 + \tan^2\alpha}$$

$$\cos 2\alpha = \frac{\cos^2\alpha - \sin^2\alpha}{1} = \frac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha + \sin^2\alpha} = \frac{\cancel{\cos^2\alpha}\left(1 - \frac{\sin^2\alpha}{\cos^2\alpha}\right)}{\cancel{\cos^2\alpha}\left(1 + \frac{\sin^2\alpha}{\cos^2\alpha}\right)}$$

$$= \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha}$$

These identities are often used to integrate rational functions by trigonometric substitutions.

5.4 Equations

5.66. Equations that include periodic functions have periodic solution set. Sinusoidal function crosses the horizontal axis every integer multiple of $\pm\pi$ starting with $x = 0$, that is to say

$$\sin x = 0 \quad \therefore \quad x = n\pi, \quad \text{where,} \quad n = 0, \pm 1, \pm 2, \ldots$$

5.67. Equations that include periodic functions have periodic solution set. Co-sinusoidal function crosses the horizontal axis every integer multiple of $\pm\pi$ starting with $x = \pi/2$, that is to say

$$\cos x = 0 \quad \therefore \quad x = (2k+1)\frac{\pi}{2}, \quad \text{where,} \quad k = 0, \pm 1, \pm 2, \ldots$$

5.68. Solving trigonometric equations is often done by deriving factorized form of the equation (similar to P. 5.63); then by exploiting the general property: if $A \times B \times C = 0$, then either $A = 0$ or $B = 0$ or $C = 0$. Thus,

$$\sin x = \sin\alpha \quad \therefore \quad \sin x - \sin\alpha = 0 \quad \therefore$$

$$\left\{\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right\}$$

$$2\cos\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right) = 0$$

Therefore,

$$\cos\left(\frac{x+\alpha}{2}\right) = 0 \quad \therefore \quad \frac{x+\alpha}{2} = (2n+1)\frac{\pi}{2} \quad \therefore \quad x = -\alpha + (2n+1)\pi$$

$$\sin\left(\frac{x-\alpha}{2}\right) = 0 \quad \therefore \quad \frac{x-\alpha}{2} = k\pi \quad \therefore \quad x = \alpha + 2k\pi$$

where $(k, n) = 0, \pm 1, \pm 2, \ldots$ but necessarily at the same time.

5.69. By exploiting the equivalence relationship between $\sin x$ and $\cos x$ functions (i.e., their argument difference of $\pi/2$), their equality may be resolved as

$$\sin x = \cos x \quad \therefore \quad \cos\left(x - \frac{\pi}{2}\right) = \cos x \quad \therefore \quad x - \frac{\pi}{2} = \pm x + 2n\pi$$

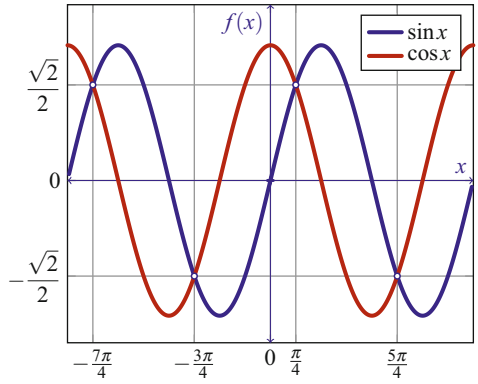
because x takes both positive and negative signs. Therefore, there are two possible solutions that must be reviewed as follows:

$$\underline{\text{case '}+x\text{':}} \quad \therefore \quad \cancel{x} - \frac{\pi}{2} \neq \cancel{x} + 2n\pi \quad \therefore \quad \text{no solution}$$

$$\underline{\text{case '}-x\text{':}} \quad \therefore \quad x - \frac{\pi}{2} = -x + 2n\pi \quad \therefore \quad 2x = 2n\pi + \frac{\pi}{2} \quad \therefore \quad x = n\pi + \frac{\pi}{4}$$

where $n = 0, \pm 1, \pm 2, \ldots$ For example, within the interval $n = -2, -1, 0, 1$ the solution set

Fig. 5.9 Example P.5.69



$$x = n\pi + \frac{\pi}{4} \quad \therefore \quad \begin{cases} n = -2 & \therefore \quad x = -2\pi + \dfrac{\pi}{4} = -\dfrac{8\pi}{4} + \dfrac{\pi}{4} = -\dfrac{7\pi}{4} \\ n = -1 & \therefore \quad x = -\pi + \dfrac{\pi}{4} = -\dfrac{4\pi}{4} + \dfrac{\pi}{4} = -\dfrac{3\pi}{4} \\ n = 0 & \therefore \quad x = \dfrac{\pi}{4} \\ n = 1 & \therefore \quad x = \pi + \dfrac{\pi}{4} = \dfrac{4\pi}{4} + \dfrac{5\pi}{4} = \dfrac{5\pi}{4} \end{cases}$$

is shown in Fig. 5.9. Both $\sin(x)$ and $\cos(x)$ equal to $\pm\sqrt{2}/2$ at these points.

5.70. After converting into factorized form of this equation, it follows that

$$\sin x = \sin 2x \quad \therefore \quad \sin 2x - \sin x = 0$$

$$\left\{ \sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \right\}$$

$$2\cos\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right) = 0$$

$$\therefore$$

$$(1) \quad \cos\left(\frac{3x}{2}\right) = 0 \quad \therefore \quad \frac{3x}{2} = (2n+1)\frac{\pi}{2} \quad \therefore \quad \underline{x = (2n+1)\frac{\pi}{3}}$$

$$(2) \quad \sin\left(\frac{x}{2}\right) = 0 \quad \therefore \quad \frac{x}{2} = k\pi \quad \therefore \quad \underline{x = 2k\pi}$$

where $n, k = 0, \pm1, \pm2, \ldots$ For example, within the interval $n = -2, -1, 0, 1, 2$ the solution set

$$x = 2k\pi \quad \therefore \quad \begin{cases} k = -1 & \therefore \quad x = -2\pi \\ n = 0 & \therefore \quad x = 0 \\ n = 1 & \therefore \quad x = 2\pi \end{cases}$$

as well as,

$$x = (2n+1)\,\frac{\pi}{3} \quad \therefore \quad \begin{cases} n=-2 & \therefore \quad x = (-4+1)\,\dfrac{\pi}{3} = -\pi \\ n=-1 & \therefore \quad x = (-2+1)\,\dfrac{\pi}{3} = -\dfrac{\pi}{3} \\ n=0 & \therefore \quad x = (0+1)\,\dfrac{\pi}{3} = \dfrac{\pi}{3} \\ n=1 & \therefore \quad x = (2+1)\,\dfrac{\pi}{3} = \pi \\ n=2 & \therefore \quad x = (4+1)\,\dfrac{\pi}{3} = \dfrac{5\pi}{3} \end{cases}$$

The two solution sets where $\sin x = \sin 2x$ are shown in Fig. 5.10 (left). In addition, equation

$$f(x) = \sin x - \sin 2x = 0$$

is shown explicitly in Fig. 5.10 (right) with the same solution set.

5.71. Using the same idea as in P. 5.68 to P. 5.70, factorized form of this equation is

$$2\sin^2 x + \sin x = 0 \quad \therefore \quad (2\sin x + 1)\sin x = 0$$
$$\therefore \quad (2\sin x + 1) = 0 \quad \text{or} \quad \sin x = 0$$

Therefore,

Equation 1: $\sin x = 0$

$$\sin x = 0 \quad \therefore \quad \underline{x = n\pi} \quad \text{where} \quad n = 0, \pm 1, \pm 2, \ldots$$

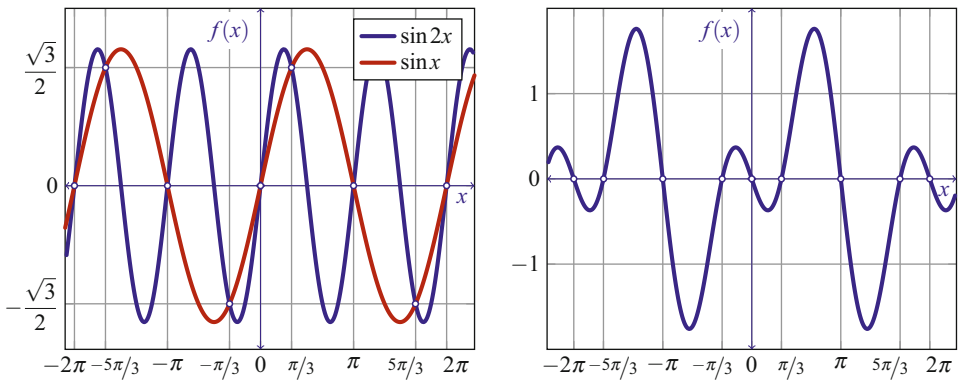


Fig. 5.10 Example P.5.70

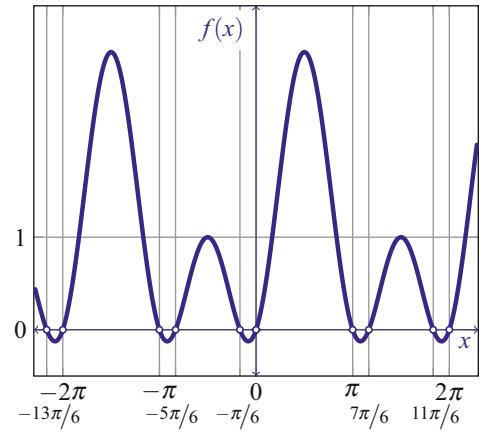


Fig. 5.11 Example P.5.71

Equation 2: $(2\sin x + 1) = 0$

$$(2\sin x + 1) = 0 \;\;\therefore\;\; \sin x + \frac{1}{2} = 0 \;\;\therefore\;\; \sin x + \sin\left(\frac{\pi}{6}\right) = 0 \;\;\therefore$$
$$2\sin\left(\frac{x+\pi/6}{2}\right)\cos\left(\frac{x-\pi/6}{2}\right) = 0$$

Again, there are two equations to solve.

1. Roots of sin function are found at multiples of π, i.e., $0, \pm\pi, \pm 2\pi, \ldots$

$$\sin\left(\frac{x+\pi/6}{2}\right) = 0 \;\;\therefore\;\; \frac{x+\pi/6}{2} = k\pi \;\;\therefore\;\; \underline{x = -\frac{\pi}{6} + 2k\pi} \quad \text{where} \quad k = 0, \pm 1, \pm 2, \ldots$$

2. Roots of cos function are at odd multiples of $\pi/2$, i.e., $\pm(2m+1)\pi/2$

$$\cos\left(\frac{x-\pi/6}{2}\right) = 0 \;\;\therefore\;\; \frac{x-\pi/6}{2} = (2m+1)\frac{\pi}{2} \;\;\therefore\;\; \underline{x = (2m+1)\pi + \frac{\pi}{6}}$$

where $m = 0, \pm 1, \pm 2, \ldots$. The complete solution set is summarized in graph of $f(x) = 2\sin^2 x + \sin x$ (Fig. 5.11). One stream of the solutions (i.e., the intersect points) is generated for $n = 0, \pm 1, \pm 2, \ldots$, second for $k = 0, \pm 1, \pm 2, \ldots$, and third for $m = 0, \pm 1, \pm 2, \ldots$.

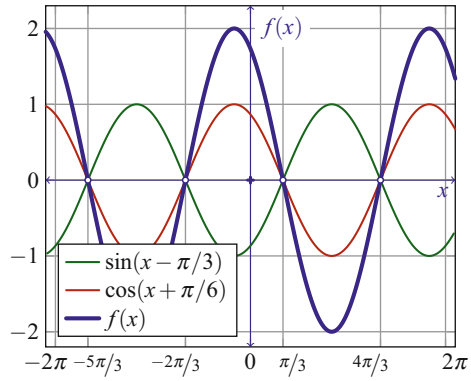
5.72. Periodicity of $\tan x = \sin x / \cos x$ function is periodic by $n\pi$; thus, we can solve this equation as follows:

$$\cos(x + \pi/6) = \sin(x - \pi/3)$$

$$\left\{\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\end{aligned}\right\}$$

(cont.)

Fig. 5.12 Example P.5.72



$$\cos(x)\cos\left(\frac{\pi}{6}\right) - \sin(x)\sin\left(\frac{\pi}{6}\right) = \sin(x)\cos\left(\frac{\pi}{3}\right) - \cos(x)\sin\left(\frac{\pi}{3}\right)$$

$$\cos(x)\frac{\sqrt{3}}{2} - \sin(x)\frac{1}{2} = \sin(x)\frac{1}{2} - \cos(x)\frac{\sqrt{3}}{2}$$

$$2\sqrt{3}\cos x = 2\sin x \quad \therefore \quad \tan x = \frac{\sqrt{3}}{1} = \frac{\sqrt{3}/2}{1/2} = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sin(x)}{\cos(x)}$$

Both $\sin(x)$ and $\cos(x)$ cross the horizontal axis with the periodicity of π; therefore

$$x = \frac{\pi}{3} + n\pi \quad \text{where} \quad n = 0, \pm 1, \pm 2, \ldots$$

The solution set of $\cos(x + \pi/6) = \sin(x - \pi/3)$ equation is illustrated in Fig. 5.12 to show the intersecting points, as well as $f(x) = \cos(x + \pi/6) - \sin(x - \pi/3)$.

5.73. With the help of algebra, identities, and unit circle, it follows that

$$\tan\alpha = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2 = \sqrt{3}\sqrt{2} + \sqrt{3} - \sqrt{2} - (\sqrt{2})^2$$

$$= \sqrt{3}\left(\sqrt{2}+1\right) - \sqrt{2}\left(\sqrt{2}+1\right) = \frac{\sqrt{2}-1}{\sqrt{2}-1}\left(\sqrt{2}+1\right)\left(\sqrt{3}-\sqrt{2}\right)$$

$$\left\{(a-b)(a+b) = a^2 - b^2\right\}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}\frac{2}{2} = \frac{\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2} - \frac{1}{2}} = \frac{\sin 60° - \sin 45°}{\sin 45° - \sin 30°}$$

$$\left\{\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right\}$$

$$= \frac{2\cos\left(\frac{105°}{2}\right)\sin\left(\frac{15°}{2}\right)}{2\cos\left(\frac{75°}{2}\right)\sin\left(\frac{15°}{2}\right)} = \frac{\cos\left(\frac{(180-75)°}{2}\right)}{\cos\left(\frac{75°}{2}\right)} = \frac{\cos\left(90° - \frac{75°}{2}\right)}{\cos\left(\frac{75°}{2}\right)}$$

(cont.)

$$\left\{\cos(90^\circ - \alpha) = \sin\alpha\right\}$$

$$= \frac{\sin\left(\frac{75^\circ}{2}\right)}{\cos\left(\frac{75^\circ}{2}\right)} = \tan\left(\frac{75^\circ}{2}\right) \quad \therefore \quad \alpha = \frac{75^\circ}{2} = 37.5^\circ$$

which is in the I quadrant.

5.74. Factorized form is derived as

$$\cos\left(\frac{\pi}{3} - x\right) = \cos\left(\frac{\pi}{6}\right)$$

$$\cos\left(\frac{\pi}{3} - x\right) - \cos\left(\frac{\pi}{6}\right) = 0$$

$$\left\{\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right\}$$

$$-2 \ \sin\left(\frac{\frac{\pi}{3} - x + \frac{\pi}{6}}{2}\right) \ \sin\left(\frac{\frac{\pi}{3} - x - \frac{\pi}{6}}{2}\right) = 0$$

$$\underbrace{-2}_{A} \ \underbrace{\sin\left(\frac{\pi - 2x}{4}\right)}_{B} \ \underbrace{\sin\left(\frac{\pi - 6x}{12}\right)}_{C} = 0$$

Obviously, $-2 \neq 0$ leaving either $B = 0$ or $C = 0$ terms to be zero, where

$$\text{if, } \sin\theta = 0 \quad \therefore \quad \theta = n\pi, \ n = 0, \pm1, \pm2, \ldots$$

Therefore, within the interval $0 \leq x < 2\pi$,

$\underline{n = 0:}$

$$B: \quad \frac{\pi - 2x}{4} = 0 \quad \therefore \quad x = \frac{\pi}{2} \in [0, 2\pi] \ \checkmark$$

$$C: \quad \frac{\pi - 6x}{12} = 0 \quad \therefore \quad x = \frac{\pi}{6} \in [0, 2\pi] \ \checkmark$$

$\underline{n = 1:}$

$$B: \quad \frac{\pi - 2x}{4} = \pi \quad \therefore \quad x = -\frac{3\pi}{2} < 0 \ \times$$

$$C: \quad \frac{\pi - 6x}{12} = \pi \quad \therefore \quad x = -\frac{11\pi}{6} < 0 \ \times$$

$\underline{n = -1:}$

$$B: \quad \frac{\pi - 2x}{4} = -\pi \quad \therefore \quad x = \frac{5\pi}{2} > 2\pi \quad \times$$
$$C: \quad \frac{\pi - 6x}{12} = -\pi \quad \therefore \quad x = \frac{13\pi}{6} > 2\pi \quad \times$$

In conclusion, within the interval $0 < x \leq 2\pi$, there are only two solutions (see Fig. 5.13).

$$x = \left\{\frac{\pi}{6}, \frac{\pi}{2}\right\}$$

5.75. Similarly to P. 5.74,

$$\sin(3x) = \sin\left(x - \frac{\pi}{2}\right)$$
$$\sin(3x) - \sin\left(x - \frac{\pi}{2}\right) = 0$$
$$\left\{\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right\}$$
$$2\ \cos\left(\frac{3x + x - \frac{\pi}{2}}{2}\right)\ \sin\left(\frac{3x - x + \frac{\pi}{2}}{2}\right) = 0$$
$$\underbrace{2}_{A}\ \underbrace{\cos\left(\frac{8x - \pi}{4}\right)}_{B}\ \underbrace{\sin\left(\frac{4x + \pi}{4}\right)}_{C} = 0$$

Obviously, $2 \neq 0$ leaving either $B = 0$ or $C = 0$ terms to be zero. Term C is a $\sin\theta$ function, and B is in the form of $\cos\theta$, where

$$\text{if, } \cos\theta = 0 \quad \therefore \quad \theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots \quad \text{in general:} \quad \theta = \frac{\pi}{2} \pm n\pi, \ n = 0, 1, 2, \ldots$$
$$\text{if, } \sin\theta = 0 \quad \therefore \quad \theta = n\pi, \ n = 0, \pm1, \pm2, \ldots$$

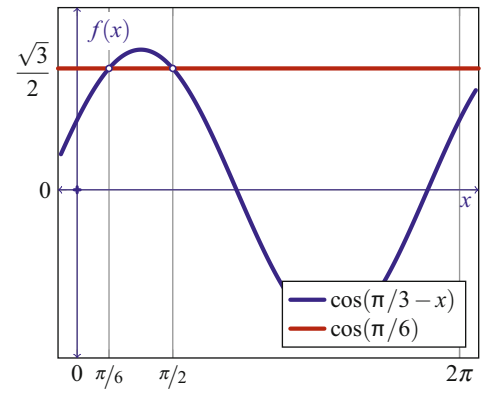


Fig. 5.13 Example P.5.74

Therefore, within the interval $-\pi < x \leq \pi$,

$\underline{n = 0:}$

$$B: \quad \frac{8x - \pi}{4} = \frac{\pi}{2} \quad \therefore \quad x = \frac{3\pi}{8} \in [-\pi, \pi] \checkmark$$

$$C: \quad \frac{4x + \pi}{4} = 0 \quad \therefore \quad x = -\frac{\pi}{4} \in [-\pi, \pi] \checkmark$$

$\underline{n = 1:}$

$$B: \quad \frac{8x - \pi}{4} = \frac{\pi}{2} + \pi \quad \therefore \quad x = \frac{7\pi}{8} \in [-\pi, \pi] \checkmark$$

$$C: \quad \frac{4x + \pi}{4} = \pi \quad \therefore \quad x = \frac{3\pi}{4} \in [-\pi, \pi] \checkmark$$

$\underline{n = -1:}$

$$B: \quad \frac{8x - \pi}{4} = \frac{\pi}{2} - \pi \quad \therefore \quad x = -\frac{\pi}{8} \in [-\pi, \pi] \checkmark$$

$$C: \quad \frac{4x + \pi}{4} = -\pi \quad \therefore \quad x = -\frac{5\pi}{4} < -\pi \times$$

$\underline{n = 2:}$

$$B: \quad \frac{8x - \pi}{4} = \frac{\pi}{2} + 2\pi \quad \therefore \quad x = \frac{11\pi}{8} > \pi \times$$

$$C: \quad \frac{4x + \pi}{4} = 2\pi \quad \therefore \quad x = \frac{7\pi}{4} > \pi \times$$

$\underline{n = -2:}$

$$B: \quad \frac{8x - \pi}{4} = \frac{\pi}{2} - 2\pi \quad \therefore \quad x = -\frac{5\pi}{8} \in [-\pi, \pi] \checkmark$$

$$C: \quad \frac{4x + \pi}{4} = -2\pi \quad \therefore \quad x = -\frac{9\pi}{4} < -\pi \times$$

In conclusion, within the interval $-\pi < x \leq \pi$, there are six solutions (see Fig. 5.14).

$$x = \left\{ -\frac{5\pi}{8}, -\frac{2\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{6\pi}{8}, \frac{7\pi}{8} \right\}$$

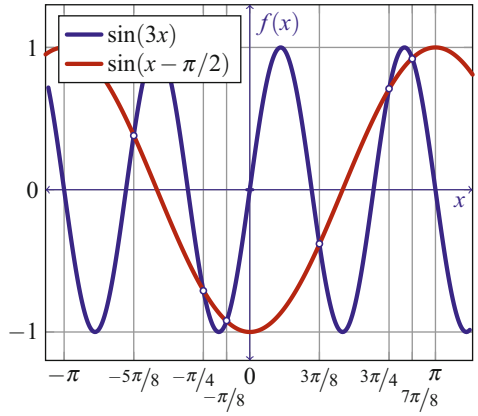


Fig. 5.14 Example P.5.75

5.5 Inequalities

5.76. A simple inequality may be solved by the use of unit circle as follows:

$$2\cos x + 1 < 0$$

$$\cos x < -\frac{1}{2}$$

By inspection of unit circle (see Fig. 5.1), boundaries of the solution set are defined by equality, i.e.,

$$x_1 = \frac{4\pi}{3} \pm 2n\pi \quad \text{and} \quad x_2 = -\frac{4\pi}{3} \pm 2n\pi$$

where, for a simple $\cos(x)$ function, periodicity is 2π. It is evident from the unit circle that $\cos x$ (i.e., the horizontal projection length) is less than "$-1/2$" within the interval (note, the negative side)

$$-\frac{4\pi}{3} < x < \frac{4\pi}{3}$$

$$\therefore \quad \text{or, equivalently}$$

$$x < -\pi \pm \frac{\pi}{3}$$

repeated every $2n\pi$ where $n = 0, \pm 1, \pm 2, \ldots$, see Fig. 5.15.

5.77. Similarly to P. 5.76,

$$2\sin x - \sqrt{2} > 0$$

$$\sin x > \frac{\sqrt{2}}{2}$$

this inequality is satisfied for

Fig. 5.15 Example P.5.76

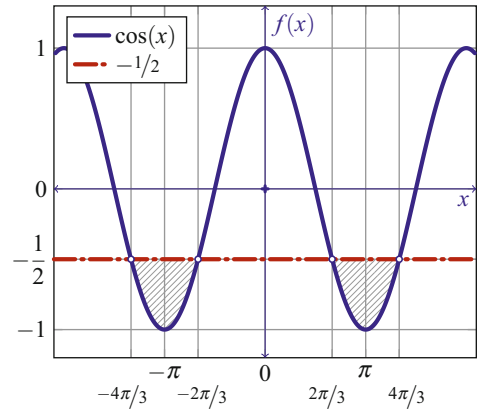
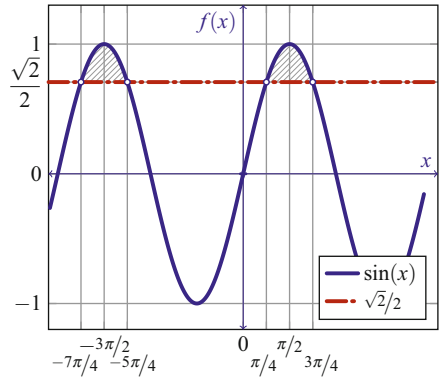
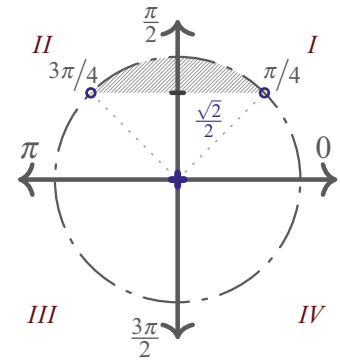


Fig. 5.16 Example P.5.77





$$\frac{\pi}{6} + 2n\pi < x < \frac{5\pi}{6} + 2n\pi \quad \text{where,} \quad n = 0, \pm 1, \pm 2, \ldots$$

as illustrated in Fig. 5.16. The interval solution sets are marked by shaded areas that are periodic in Fig. 5.16 (left). This periodicity may be equivalently illustrated by unit circle in Fig. 5.16 (right) where, due to periodicity, the solution sets are superimposed above $\sin x = \sqrt{2}/2$ value (which is vertical projection for $\sin x$) and marked by shaded area.

5.78. More examples of the same idea as in P. 5.66 to P. 5.77.

$$\cos x - \sin x < 1 \quad \Big| \times \frac{\sqrt{2}}{2}$$

$$\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x < \frac{\sqrt{2}}{2}$$

$$\sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x < \frac{\sqrt{2}}{2}$$

$$\left\{\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha - \beta)\right\}$$

$$\sin\left(\frac{\pi}{4} - x\right) < \frac{\sqrt{2}}{2}$$

By inspection of unit circle, see Fig. 5.1, it is evident that

$$\sin x = \frac{\sqrt{2}}{2} \Rightarrow x = \frac{\pi}{4} \quad \text{or,} \quad x = -\frac{5\pi}{4}$$

Therefore,

$$\frac{\pi}{4} - x < \frac{\pi}{4} \quad \therefore \quad x > 0 + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots \quad \text{and}$$
$$\frac{\pi}{4} - x > -\frac{5\pi}{4} \quad \therefore \quad x < \frac{3\pi}{2} + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

The solution set is illustrated in Fig. 5.17 (left). Alternatively, the solution set may be shown in unit circle, where the periodic nature of the solution set is more evident, as intervals of $0 < x < 3\pi/2$ and $-2\pi < x - \pi/2$ are overlapped and repeated with $2pi$ periodicity. Note that, due to strict inequality, boundary points are not included in the solution set, i.e., $x \neq 2n\pi$ and $x \neq 3\pi/2$.

5.79. Note that $\sin(x) = \sin(x + 2\pi)$, i.e., it is periodic by $T = 2\pi$. However, multiplied argument reduces proportionally period, that is to say $\sin(3x)$ is periodic with $T = 2\pi/3$. Given inequality may be then resolved as follows:

$$\sin 3x - \frac{\sqrt{3}}{2} \geq 0$$
$$\sin 3x - \sin\left(\frac{\pi}{3}\right) \geq 0$$
$$\left\{\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right\}$$
$$2\cos\left(\frac{3x + \frac{\pi}{3}}{2}\right)\sin\left(\frac{3x - \frac{\pi}{3}}{2}\right) \geq 0$$
$$2\cos\left(\frac{9x+\pi}{6}\right)\sin\left(\frac{9x-\pi}{6}\right) \geq 0$$

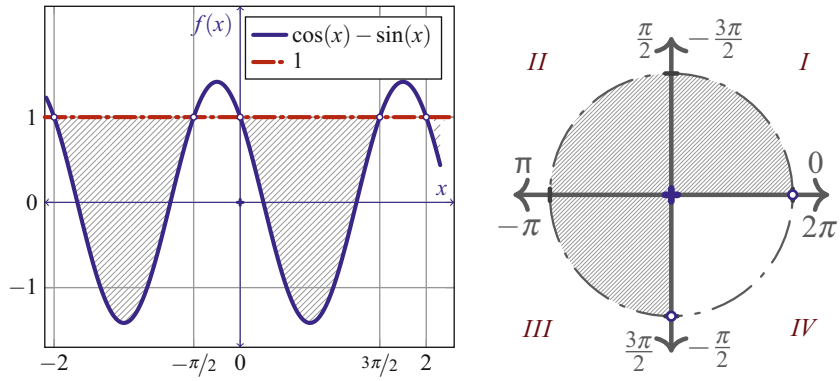


Fig. 5.17 Example P.5.78

Therefore, as per unit circle, if $\cos x \geq 0 \;\; \therefore \;\; x \leq \pm\pi/2$ and if $\sin \geq 0 \;\; \therefore \;\; 0 \leq x \leq \pi$, then

$$\cos\left(\frac{9x+\pi}{6}\right) \geq 0 \;\; \therefore \;\; -\frac{\pi}{2} \leq \frac{9x+\pi}{6} \leq \frac{\pi}{2} \;\; \therefore \;\; -\frac{4\pi}{9} \leq x \leq \frac{2\pi}{9}$$

and,

$$\sin\left(\frac{9x-\pi}{6}\right) \geq 0 \;\; \therefore \;\; 0 \leq \frac{9x-\pi}{6} \leq \pi \;\; \therefore \;\; \frac{\pi}{9} \leq x \leq \frac{7\pi}{9}$$

The two intervals are overlapping if $\pi/9 \leq x \leq 2\pi/9$ (see the upper diagram in Fig. 5.18 (left)).

The second possibility is if $\cos x \leq 0 \;\; \therefore \;\; \pi/2 \leq x \leq 3\pi/2$ and if $\sin \leq 0 \;\; \therefore \;\; -\pi \leq x \leq 0$, then

$$\cos\left(\frac{9x+\pi}{6}\right) \leq 0 \;\; \therefore \;\; \frac{\pi}{2} \leq \frac{9x+\pi}{6} \leq \frac{3\pi}{2} \;\; \therefore \;\; \frac{2\pi}{9} \leq x \leq \frac{8\pi}{9}$$

and,

$$\sin\left(\frac{9x-\pi}{6}\right) \leq 0 \;\; \therefore \;\; -\pi \leq \frac{9x-\pi}{6} \leq 0 \;\; \therefore \;\; -\frac{5\pi}{9} \leq x \leq \frac{\pi}{9}$$

Therefore, the two intervals are *not* overlapping, i.e., the solution set is empty (see the lower diagram in Fig. 5.18 (left)). In summary, including the periodicity of $T = 2n\pi/3$ the solution set is

$$\frac{\pi}{9} \pm \frac{6n\pi}{9} \leq x \leq \frac{2\pi}{9} \pm \frac{6n\pi}{9}$$

where, $n = 0, \pm 1, \pm 2, \ldots$, see Fig. 5.18 (right).

5.80. Inequalities that include absolute functions are equivalent to two inequalities, thus

$$|\sin x| \geq \frac{1}{2} \;\; \therefore \;\; -\frac{1}{2} \geq \sin x \geq \frac{1}{2}$$

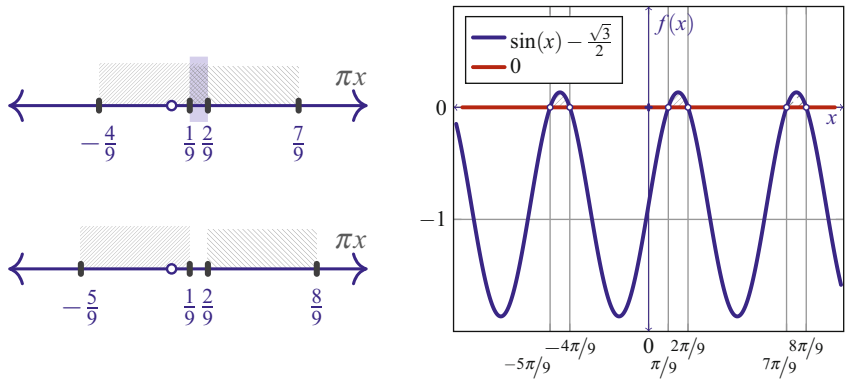


Fig. 5.18 Example P.5.79

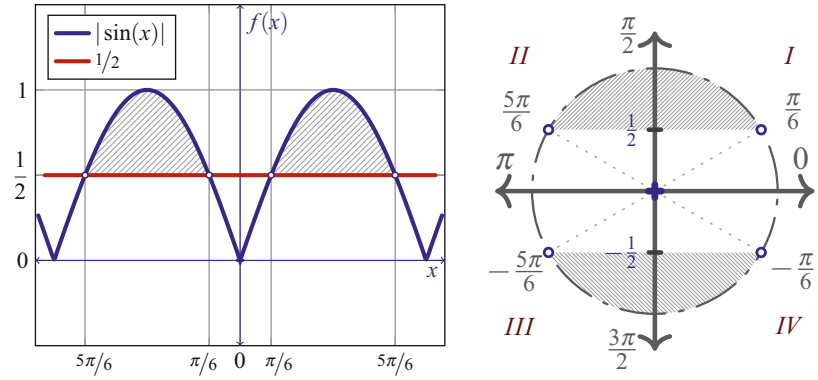


Fig. 5.19 Example P.5.80

By inspection of unit circle (see Fig. 5.1), the solution is simply

$$-\frac{5\pi}{6} + 2n\pi \le x \le -\frac{\pi}{6} + 2n\pi$$

and,

$$\frac{\pi}{6} + 2n\pi \le x \le \frac{5\pi}{6} + 2n\pi$$

where $n = 0, \pm 1, \pm 2, \ldots$ (Fig. 5.19).

5.81. Absolute value functions are decomposed into two cases.

Case 1: $\sin x > 0 \quad \therefore \quad |\sin x| = \sin x$, thus

$$\cancel{\sin x} = \cancel{\sin x} + 2\cos x \quad \therefore \quad 2\cos x = 0 \quad \therefore \quad x = \frac{\pi}{2} + 2n\pi, \quad \text{where} \quad n = 0, \pm 1, \pm 2, \ldots$$

Case 2: $\sin x < 0 \quad \therefore \quad |\sin x| = -\sin x$, thus

$$-\sin x = \sin x + 2\cos x \quad \therefore \quad 2\sin x + 2\cos x = 0 \quad \therefore \quad \sin x + \cos x = 0$$

Using same idea as in P. 5.78,

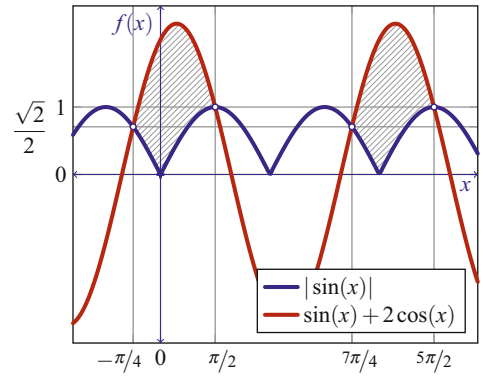
$$\sin x + \cos x = 0 \quad \Big| \times \frac{\sqrt{2}}{2}$$

$$\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x = 0$$

$$\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x = 0$$

$$\left\{\cos\alpha\sin\beta + \sin\alpha\cos\beta = \sin(\alpha+\beta)\right\}$$

Fig. 5.20 Example P.5.81



$$\therefore \quad \sin\left(x+\frac{\pi}{4}\right)=0 \quad \therefore \quad x+\frac{\pi}{4}=0$$

$$\therefore \quad x=-\frac{\pi}{4}+2n\pi, \quad \text{where} \quad n=0,\pm1,\pm2,\ldots$$

The two streams of periodic solutions are illustrated in Fig. 5.20, where the periodic solution set interval is

$$-\frac{\pi}{4}+2n\pi \le x \le \frac{\pi}{2}+2n\pi$$

Complex Algebra

6

Basic forms and identities of complex numbers:

$$
\begin{aligned}
j^2 &= -1 \\
z &= a + jb = \mathrm{Re}\,(z) + j\ \mathrm{Im}\,(z) \\
z^* &= a - jb \\
|z|^2 &= z\,z^* = (a + jb)(a - jb) = a^2 + b^2 = \mathrm{Re}\,(z)^2 + \mathrm{Im}\,(z)^2 \\
z_1 &= a + jb \quad \text{and} \quad z_2 = c + jd \quad \therefore \quad \text{if } z_1 = z_2 \Rightarrow a = c \quad \text{and} \quad b = d \\
\arg(z) &= \varphi = \mathrm{atan2}\,(\mathrm{Im}\,(z)\,, \mathrm{Re}\,(z)) = \mathrm{atan2}\,(b, a) \\
z &= |z|e^{j\varphi} = |z|\,(\cos\varphi + j\,\sin\varphi) \quad \text{Euler's formula}
\end{aligned}
$$

Note the difference between $\arctan(x)$ and $\mathrm{atan2}\,(y, x)$ functions. Geometrical interpretation of the equivalence between Pythagorean triangles, complex numbers, and vector addition is illustrated in Fig. 6.1. Note that points on unit circle in Fig. 5.1 may be interpreted as complex numbers through Euler's formula.

Problems

6.1 Basic Complex Number Forms

Calculate $|z|$ and $\arg(z)$ in P. 6.1 to P. 6.5, and comment on periodic nature of complex numbers in general:

6.1. $z = j$ **6.2.** $z = j^2$ **6.3.** $z = j^3$ **6.4.** $z = j^4$ **6.5.** $z = j^5$

Simplify expressions in P. 6.6 to P. 6.24:

6.6. $j^2 + j^3 + j^4$ **6.7.** $j^{-5} + j^{-17} + j^{36}$ **6.8.** $(1 + i)^4 + (1 - i)^4$

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023

R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41200-4_6

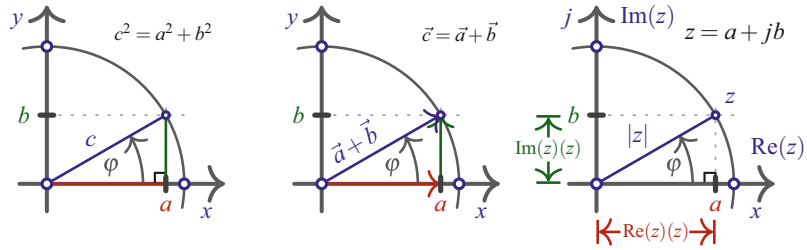


Fig. 6.1 Pythagora's theorem (unit circle), vectors, complex numbers

6.9. $j^5 + j^{-4} + j^{121}$

6.10. $(3 + 4j)(3 - 4j)$

6.11. $j^{125} + (-j)^{60} + j^{83}$

6.12. $(2j)^2 + (-2j)^{-4}$

6.13. $(1 + j)^{50}$

6.14. $(1 - j)^{100}$

6.15. $\dfrac{1}{j} + \left(\dfrac{1+j}{\sqrt{2}}\right)^4$

6.16. $\dfrac{1 + j\sqrt{3}}{2}\dfrac{1 - j\sqrt{3}}{2}$

6.17. $\left(\dfrac{4}{j\sqrt{3} - 1}\right)^{12}$

6.18. $\dfrac{(1 + j)^{1000}}{(1 - j)^{500}}$

6.19. $\dfrac{j^{102} + j^{101}}{j^{100} - j^{99}}$

6.20. $\dfrac{(j\sqrt{3} - 1)^2}{(2j)^2}$

6.21. $\left(\dfrac{1+j}{\sqrt{2}}\right)^{1000} + \left(\dfrac{1-j}{\sqrt{2}}\right)^{1000}$

6.22. $\dfrac{-41 + 63j}{50} - \dfrac{6j + 1}{1 - 7j}$

6.23. $\left(\dfrac{1 + j\sqrt{3}}{2}\right)^{6\,066} + \left(\dfrac{1 - j\sqrt{3}}{2}\right)^{6\,066}$

6.24. $\left(\dfrac{\sqrt{3}}{2} + \dfrac{j}{2}\right)^{108} + \left(\dfrac{\sqrt{12}}{4} + \dfrac{1}{2j}\right)^{216}$

6.2 Polar Forms

Convert complex forms given in P. 6.25 to P. 6.29 into their respective equivalent polar forms:

6.25. $z = i$

6.26. $z = 2 - i$

6.27. $(2 - j)^6$

6.28. $z = j^{2023}$

6.29. $z = \dfrac{(1 + j)^{200}(6 + 2j) - (1 - j)^{198}(3 - j)}{(1 + j)^{196}(23 - 7j) + (1 - j)^{194}(10 + 2j)}.$

6.3 Complex Plane

Given complex numbers in P. 6.30 to P. 6.42, in each case, show them as vectors in the complex plane:

6.30. $z_1 = 1$ **6.31.** $z_2 = -1$ **6.32.** $z_3 = j$ **6.33.** $z_4 = -j$

6.34. $z_5 = \frac{\sqrt{3}}{2} + j\frac{1}{2}$ **6.35.** $z_6 = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$ **6.36.** $z_7 = \frac{1}{2} + j\frac{\sqrt{3}}{2}$

6.37. $z_8 = -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$ **6.38.** $z_9 = -\frac{\sqrt{3}}{2} + j\frac{1}{2}$ **6.39.** $z_{10} = -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}$

6.40. $z_{11} = \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}$ **6.41.** $z_{12} = \frac{\sqrt{3}}{2} - j\frac{1}{2}$ **6.42.** $z_{13} = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$

6.4 Euler Identity

Given complex numbers in P. 6.43 to P. 6.54, convert them into their equivalent form by Euler's equation. Also, write their complex conjugate z^* pair and calculate product $z\,z^*$. Compare P. 6.30 to P. 6.42:

6.43. $z = e^{j\,0}$ **6.44.** $z = e^{j\,\frac{\pi}{6}}$ **6.45.** $z = e^{j\,\frac{\pi}{4}}$ **6.46.** $z = e^{j\,\frac{\pi}{3}}$

6.47. $z = e^{j\,\frac{\pi}{2}}$ **6.48.** $z = e^{j\,\frac{3\pi}{4}}$ **6.49.** $z = e^{j\,\frac{5\pi}{6}}$ **6.50.** $z = e^{j\,\pi}$

6.51. $z = e^{j\,\frac{5\pi}{4}}$ **6.52.** $z = e^{j\,\frac{3\pi}{2}}$ **6.53.** $z = e^{j\,\frac{7\pi}{4}}$ **6.54.** $z = e^{j\,\frac{11\pi}{6}}$

Given numbers in P. 6.55 to P. 6.62, calculate their respective $|z|$ $\arg(z)$:

6.55. $z = \frac{1}{2}\,e^{j\,\frac{\pi}{3}}$ **6.56.** $z = -2\,e^{-j\,\frac{5\pi}{6}}$ **6.57.** $z = -\frac{1}{2}\,e^{j\,\frac{7\pi}{4}}$ **6.58.** $z = \left(\sqrt{2}\,e^{j\,\frac{\pi}{4}}\right)^2$

6.59. $z = \left(3\,e^{j\,\frac{\pi}{3}}\right)^3$ **6.60.** $z = \left(4\,e^{-j\,\frac{5\pi}{6}}\right)^5$ **6.61.** $z = \left(0.2\,e^{j\,\frac{7\pi}{4}}\right)^4$ **6.62.** $z = \left(-3\,e^{-j\,\frac{\pi}{6}}\right)^3$

6.5 Rational Powers

Given complex numbers in P. 6.63 to P. 6.69, calculate all their parameters:

6.63. $z_1 = \sqrt{j}$ **6.64.** $z_2 = \sqrt{-j}$ **6.65.** $z_3 = \sqrt[3]{j}$ **6.66.** $z_4 = \sqrt[4]{j}$

6.67. $z_5 = \sqrt[5]{j}$ **6.68.** $z_6 = \sqrt{1/(2j)}$ **6.69.** $z_7 = \sqrt{(1-j)^3}$ **6.70.** $z_8 = \sqrt[3]{1/(1+j)^2}$

6.71. Given $z = (2j)^n$, sketch its trajectory as $n = 0, 1, 2, 3, \ldots$.

6.6 Complex Equations

Solve the following equations in $\mathbb{C}$:

6.72. $z^2 = j$

6.73. $z^2 = -j$

6.74. $(2+3j)x + (3+2j)y = 1$

6.75. $z^2 = 5 + 12j$

6.76. $z^4 + 8 + 8\sqrt{3}\,j = 0$

6.77. $\dfrac{1}{x+jy} = \dfrac{1}{2+j} + \dfrac{1}{-2+4j}$

Answers

6.1 Basic Complex Number Forms

Note that powers of a complex number j are periodic and in the complex plane (unit circle) may be interpreted as a rotational vector z.

6.1. Algebraic form $z = j$ is therefore identical to

$$z = 0 + 1 \times j \quad \therefore \quad \mathrm{Re}\,(z) = 0 \quad \text{and} \quad \mathrm{Im}\,(z) = +1.$$

Geometrical interpretation of a complex number modulus $|z|$ is as hypotenuse of a Pythagorean triangle (see Fig. 6.1), where the length of the horizontal cathetus equals to the $\mathrm{Re}\,(z)$ and the vertical cathetus equals to $\mathrm{Im}\,(z)$; therefore, by Pythagoras' theorem,

$$|z| = \sqrt{\mathrm{Re}\,(z)^2 + \mathrm{Im}\,(z)^2} = \sqrt{0^2 + 1^2} = 1 \quad \text{(i.e., a point on the unit circle)}.$$

By knowing two catheti of a Pythagorean triangle, argument of $\arg(z) = \varphi$ is calculated as the ratio of the vertical and horizontal catheti, i.e., from definition of $\tan\varphi$ as

$$\tan\varphi \overset{\text{def}}{=} \frac{b}{a} = \frac{\mathrm{Im}\,(z)}{\mathrm{Re}\,(z)} = \frac{1}{0} = \infty.$$

Note that calculation of φ may be ambiguous if the catheti length signs are ignored, and that is because there are two possibilities for $\arg(z)$ as

$$\arg(z) = \tan\varphi = \frac{1}{0} = \frac{\mathrm{Im}\,(z)}{\mathrm{Re}\,(z)} = \frac{\sin\varphi}{\cos\varphi} = \pm\infty \quad \therefore \quad \cos\varphi = 0 \quad \therefore \quad \varphi = \pm\frac{\pi}{2}$$

or equivalently

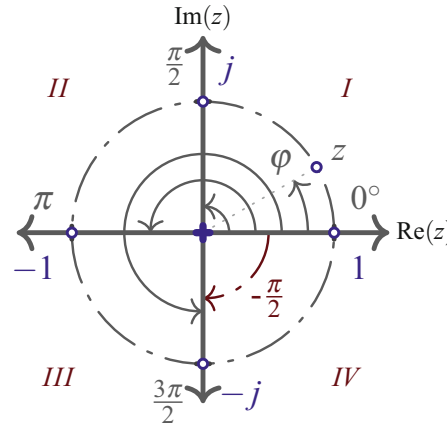
$$\tan\varphi = \pm\infty \quad \therefore \quad \varphi = \arctan(\pm\infty) = \pm\frac{\pi}{2}.$$

For that reason, keeping the catheti length signs is *mandatory*, which is resolved by using modified $\arctan(x)$ named $\mathrm{atan2}\,(b, a)$ function. Note that $\mathrm{atan2}\,(b, a)$ requires two arguments, both vertical b and horizontal a catheti, where their respective signs are explicitly included. By doing so, $\arg(z)$ are placed in the correct quadrant. That is to say,

$$\tan\varphi \overset{\text{def}}{=} \frac{\mathrm{Im}\,(z)}{\mathrm{Re}\,(z)} \quad \therefore \quad \varphi \overset{\text{def}}{=} \mathrm{atan2}\,(\mathrm{Im}\,(z)\,, \mathrm{Re}\,(z)) = \mathrm{atan2}\,(+1, 0) = \arctan\frac{+1}{0} \quad \therefore \quad \varphi = +\frac{\pi}{2}$$

which is to say that $\mathrm{Im}\,(z)$ is *positive* while $\mathrm{Re}\,(z)$ is zero, therefore positive argument. Complex plane is illustrated by unit circle where position of an arbitrary complex number z is shown with its argument φ (see Fig. 6.2). It is evident that $z = j$ is found at $\varphi = +\pi/2$, $z = -1$ at $\varphi = \pm\pi$, $z = -j$ at $\varphi = -\pi/2$, and $z = 1$ at $\varphi = 0$ rad. Note the difference between $\arg(z) = +\pi/2$ and $\arg(z) = -\pi/2$. In addition, arguments may be measured in either positive or negative direction, whichever is more convenient, for example, $\varphi = -\pi/2 \equiv +3\pi/2$.

Fig. 6.2 Example P.6.1



6.2. Algebraic form $z = j^2$ is identical to

$$z = j^2 \overset{\text{def}}{=} -1 = -1 + 0 \times j \quad \therefore \quad \text{Re}\,(z) = -1 \quad \text{and} \quad \text{Im}\,(z) = 0$$

$$\therefore$$

$$|z| = \sqrt{\text{Re}\,(z)^2 + \text{Im}\,(z)^2} = \sqrt{(-1)^2 + 0^2} = 1 \quad \text{(i.e., a point on the unit circle)}$$

$$\tan\varphi \overset{\text{def}}{=} \frac{\text{Im}\,(z)}{\text{Re}\,(z)} \quad \therefore \quad \varphi \overset{\text{def}}{=} \text{atan2}\,(\text{Im}\,(z)\,, \text{Re}\,(z)) = \text{atan2}\,(0,\ -1) \quad \therefore \quad \varphi = \pm\pi.$$

Saying that $\text{Re}\,(z) = -1$ and $\text{Im}\,(z) = 0$ is to say that in the complex plane coordinates of $z = -1$ are $(-1, 0)$; therefore, $\varphi = \pm\pi$ (see Fig. 6.2).

6.3. Algebraic form $z = j^3$ is identical to

$$z = j^3 = j^2\, j = (-1)\, j = 0 - 1 \times j \quad \therefore \quad \text{Re}\,(z) = 0 \quad \text{and} \quad \text{Im}\,(z) = -1$$

$$\therefore$$

$$|z| = \sqrt{\text{Re}\,(z)^2 + \text{Im}\,(z)^2} = \sqrt{0^2 + (-1)^2} = 1 \quad \text{(i.e., a point on the unit circle)}$$

$$\tan\varphi \overset{\text{def}}{=} \frac{\text{Im}\,(z)}{\text{Re}\,(z)} \quad \therefore \quad \varphi \overset{\text{def}}{=} \text{atan2}\,(\text{Im}\,(z)\,, \text{Re}\,(z)) = \text{atan2}\,(-1, 0) \quad \therefore \quad \varphi = -\frac{\pi}{2}.$$

Saying that $\text{Re}\,(z) = 0$ and $\text{Im}\,(z) = -1$ is to say that in the complex plane coordinates of $z = -j$ are $(0, -1)$; therefore, $\varphi = -\pi/2$ (see Fig. 6.2).

6.4. Algebraic form $z = j^4$ is identical to

$$z = j^4 = j^2\, j^2 = (-1)\,(-1) = 1 + 0 \times j \quad \therefore \quad \text{Re}\,(z) = +1 \quad \text{and} \quad \text{Im}\,(z) = 0$$

$$\therefore$$

$$|z| = \sqrt{\mathrm{Re}\,(z)^2 + \mathrm{Im}\,(z)^2} = \sqrt{1^2 + 0^2} = 1 \quad \text{(i.e., a point on the unit circle)}$$

$$\tan\varphi \overset{\text{def}}{=} \frac{\mathrm{Im}\,(z)}{\mathrm{Re}\,(z)} \quad \therefore \quad \varphi \overset{\text{def}}{=} \mathrm{atan2}\,(\mathrm{Im}\,(z)\,,\,\mathrm{Re}\,(z)) = \mathrm{atan2}\,(0,\,+1) \quad \therefore \quad \varphi = 0 \equiv \pm 2\pi.$$

Saying that $\mathrm{Re}\,(z) = +1$ and $\mathrm{Im}\,(z) = 0$ is to say that in the complex plane coordinates of $z = 1$ are $(1, 0)$; therefore, $\varphi = 0 \equiv \pm 2\pi$ (see Fig. 6.2).

6.5. Algebraic form $z = j^4$ is identical to

$$z = j^5 = j^4\, j = (1)\,(j) = 0 + 1 \times j = j$$

which is identical to P. 6.1. In conclusion, powers of j are *periodic* (therefore, equivalence with the trigonometric functions). After each full $\pm 2\pi$ rotation, a complex number z return to the initial position, as

$$\begin{aligned}
j^0 &\overset{\text{def}}{=} 1 \quad \therefore \quad \varphi = 0^\circ \quad (\pm 360^\circ) \\
j^1 &= j \quad \therefore \quad \varphi = +90^\circ \quad (-270^\circ) \\
j^2 &= -1 \quad \therefore \quad \varphi = \pm 180^\circ \\
j^3 &= -j \quad \therefore \quad \varphi = -90^\circ \quad (+270^\circ) \\
j^4 &= 1 \quad \therefore \quad \varphi = 0^\circ \quad (\pm 360^\circ) \\
j^5 &= j \quad \therefore \quad \varphi = +90^\circ \quad (-270^\circ) \\
&\vdots
\end{aligned}$$

This periodicity is conveniently exploited in practical problems involving powers of complex numbers. Recall that real numbers are subset of complex numbers (their imaginary part equals to zero), that is to say, their argument is $\varphi = 0^\circ$ for positive and $\varphi = \pm\pi$ for negative numbers.

6.6. $j^2 + j^3 + j^4 = j^2 + j\, j^2 + \left(j^2\right)^2 = -1 + j\,(-1) + (-1)^2 = -1 - j + 1 = -j$

6.7. Recall identities for powers, as well as periodicity of j^n, so that

$$\begin{aligned}
j^{-5} + j^{-17} + j^{36} &= \left\{ a^{-n} = \frac{1}{a^n},\quad a^{n\,m} = \left(a^n\right)^m,\quad a^{n+m} = a^n a^m \right\} \\
&= \frac{1}{j^5} + \frac{1}{j^{17}} + j^{36} = \frac{1}{j^{1+4}} + \frac{1}{j^{1+4\times 4}} + j^{4\times 9} \\
&= \frac{1}{j\, j^4} + \frac{1}{j\, \left(j^4\right)^4} + \left(j^4\right)^9 = \left\{ j^4 = 1,\quad 1^n = 1 \right\} \\
&= \frac{1}{j} + \frac{1}{j\,(1)^4} + (1)^9 = \frac{2}{j} + 1 = \frac{j}{j}\,\frac{2}{j} + 1 = 1 - 2j.
\end{aligned}$$

Note that $\mathrm{Re}(z) = 1; \mathrm{Im}(z) = -2$, which is in IV quadrant; $|1 - 2j| = \sqrt{5}$; and $\arg(z) = \mathrm{atan2}(-2, 1)$ ($\approx -63.43° \approx -1.107\mathrm{rad}$).

6.8. $$\begin{aligned}(1+j)^4 + (1-j)^4 &= \left((1+j)^2\right)^2 + \left((1-j)^2\right)^2 = \left\{(a \pm b)^2 = a^2 \pm 2ab + b^2\right\} \\ &= (1 + 2j - 1)^2 + (1 - 2j - 1)^2 = 4j^2 + 4j^2 = \left\{j^2 = -1\right\} \\ &= -4 - 4 = -8.\end{aligned}$$

Note that $\mathrm{Re}(z) = -8; \mathrm{Im}(z) = 0$, which is negative part of the horizontal axis; $|-8| = 8$; and $\arg(z) = \mathrm{atan2}(0, -8) = \pm\pi$.

6.9. $$j^5 + j^{-4} + j^{121} = j^{1+4} + \frac{1}{j^{1+4}} + j^{1+4\times 30} = j\,j^4 + \frac{1}{j^4} + j\left(j^4\right)^{30} = j + 1 + j = 1 + 2j.$$

Note that $\mathrm{Re}(z) = 1; \mathrm{Im}(z) = 2$, which is in I quadrant; $|1 + 2j| = \sqrt{5}$; and $\arg(z) = \mathrm{atan2}(2, 1)$ ($\approx 63.43° \approx 1.107\mathrm{rad}$).

6.10. $$(3 + 4j)(3 - 4j) = 3^2 - (4j)^2 = 9 + 16 = 25$$

Note that product of complex conjugate numbers is a real number and that $\mathrm{Re}(z) = 25; \mathrm{Im}(z) = 0$, which is positive part of the horizontal axis; $|25| = 25$; and $\arg(z) = \mathrm{atan2}(0, 25) = 0°$.

6.11. $$\begin{aligned}j^{125} + (-j)^{60} + j^{83} &= j^{1+4\times 31} + (-1)^{60} j^{4\times 15} + j^{3+4\times 20} \\ &= \left\{-1^n = -1 \quad \text{if } n \text{ is odd , and} \quad -1^n = +1 \text{ if } n \text{ is even}\right\} \\ &= j\left(j^4\right)^{31} + \left(j^4\right)^{15} + j\,j^2\left(j^4\right)^{20} \\ &= j + 1 - j = 1.\end{aligned}$$

Note that $\mathrm{Re}(z) = 1; \mathrm{Im}(z) = 0$, which is a positive part of the horizontal axis; $|1| = 1$; and $\arg(z) = \mathrm{atan2}(0, 1) = 0°$.

6.12. $$(2j)^2 + (-2j)^{-4} = -4 + \frac{1}{(-1)^4(2)^4(j)^4} = -4 + \frac{1}{16} = -\frac{63}{16}.$$

Note that $\mathrm{Re}(z) = -63/16; \mathrm{Im}(z) = 0$, which is negative part of the horizontal axis; $|-63/16| = 63/16$; and $\arg(z) = \mathrm{atan2}(0, -63/16) = \pm\pi$.

6.13. $$(1+j)^{50} = (1+j)^{2\times 25} = (2j)^{25} = 2^{25} \times j \times j^{4\times 6} = 2^{25}\,j$$

Note that $\mathrm{Re}(z) = 0; \mathrm{Im}(z) = 2^{25}$, which is positive part of the vertical axis; $|2^{25}| = 2^{25}$; and $\arg(z) = \mathrm{atan2}\left(2^{25}, 0\right) = +\pi/2$.

6.14. $$\begin{aligned}(1-j)^{100} &= (1-j)^{2\times 50} = \left((1-j)^2\right)^{50} = \left(1 - 2j - 1\right)^{50} = (-2j)^{50} \\ &= (-1)^{50}(2)^{50}(j)^{2+4\times 12} = 2^{50} \times j^2 \times \left(j^4\right)^{12} = -2^{50}.\end{aligned}$$

Note that $\mathrm{Re}(z) = -2^{50}; \mathrm{Im}(z) = 0$, which is negative part of the horizontal axis; $|-2^{50}| = 2^{50}$; and $\arg(z) = \mathrm{atan2}\left(0, -2^{50}\right) = \pm\pi$.

6.15. $\dfrac{1}{j}+\left(\dfrac{1+j}{\sqrt{2}}\right)^4=\dfrac{j}{j}\dfrac{1}{j}+\dfrac{\left((1+j)^2\right)^2}{2^2}=-j+\dfrac{(2j)^2}{4}=-j-\dfrac{4}{4}=-1-j.$

Note that $\mathrm{Re}\,(z)=-1$; $\mathrm{Im}\,(z)=-1$, which is in III quadrant; $|-1-j|=\sqrt{2}$; and $\arg(z)=\mathrm{atan2}\,(-1,-1)=-135^\circ\equiv+^{5\pi}/_4$.

6.16. $\dfrac{1+j\sqrt{3}}{2}\dfrac{1-j\sqrt{3}}{2}=\left\{(a+b)(a-b)=a^2-b^2\right\}=\dfrac{1^2-(j\sqrt{3})^2}{4}=\dfrac{1+3}{4}=1$

Note that $\mathrm{Re}\,(z)=1$; $\mathrm{Im}\,(z)=0$, which is positive part of the horizontal axis; $|1|=1$; and $\arg(z)=\mathrm{atan2}\,(0,1)=0^\circ$.

6.17. Note that

$$\arg\left(-1+j\sqrt{3}\right)\overset{\text{def}}{=}\mathrm{atan2}\left(\sqrt{3},-1\right)=\left\{\text{the special angle (see Fig. 5.5 (right))}\right\}=\frac{2\pi}{3}$$

which is in the II quadrant. It is very convenient because if multiplied by "3" this angle equals to $2\pi\equiv0^\circ$, which is a phase of a real number. What is more, in the context of complex numbers, powers force complex numbers to rotate by multiplying its phase. Therefore,

$$\begin{aligned}\left(\frac{4}{j\sqrt{3}-1}\right)^{12}&=\frac{4^{12}}{(-1+j\sqrt{3})^{12}}=\frac{4^{12}}{(-1+j\sqrt{3})^{3\times4}}=\frac{4^{12}}{\left((-1+j\sqrt{3})^2\,(-1+j\sqrt{3})\right)^4}\\&=\frac{4^{12}}{\left((-1+j\sqrt{3})^2\,(-1+j\sqrt{3})\right)^4}=\frac{4^{12}}{\left((-2-2j\sqrt{3})\,(-1+j\sqrt{3})\right)^4}\\&=\frac{4^{12}}{\left(2-2j\sqrt{3}+2j\sqrt{3}+6\right)^4}=\left\{4=2^2,\ 8=2^3\right\}=\frac{\left(2^2\right)^{12}}{\left(2^3\right)^4}\\&=2^{12}=4\,096.\end{aligned}$$

Note that $\mathrm{Re}\,(z)=4\,096$; $\mathrm{Im}\,(z)=0$, which is positive part of the horizontal axis; $|4096|=4096$; and $\arg(z)=\mathrm{atan2}\,(0,4096)=0^\circ$.

6.18. $\dfrac{(1+j)^{1000}}{(1-j)^{500}}=\dfrac{(1+j)^{2\times500}}{(1-j)^{2\times250}}=\dfrac{(2j)^{500}}{(-2j)^{250}}=\dfrac{(2j)^{500}}{\cancel{(-1)^{250}}\,(2j)^{250}}=(2j)^{250}=2^{250}\,j^{2\times125}$

$$=2^{250}\,(-1)^{125}=-2^{250}$$

Note that $\mathrm{Re}\,(z)=-2^{250}$; $\mathrm{Im}\,(z)=0$, which is negative part of the horizontal axis; $|-2^{250}|=2^{250}$; and $\arg(z)=\mathrm{atan2}\left(0,-2^{250}\right)=\pm\pi$.

6.19. As an illustration, here are two methods to simplify this complex number form:

Method 1:

$$\frac{j^{102}+j^{101}}{j^{100}-j^{99}}=\frac{j^2\,j^{4\times25}+j\,j^{4\times25}}{j^{4\times25}-j^3\,j^{4\times24}}=\frac{j^2+j}{1-j^3}=\left\{z\,z^*=(a+ib)(a-ib)=a^2+b^2=|z|^2\right\}$$

$$= \frac{-1+j}{1+j}\,\frac{1-j}{1-j} = \frac{-1+j+j+1}{1+1} = \frac{2j}{2} = j.$$

Method 2:

$$\frac{j^{102}+j^{101}}{j^{100}-j^{99}} = \frac{j^{99}\left(j^{3}+j^{2}\right)}{j^{99}(j-1)} = \frac{-1-j}{-1+j}\,\frac{-1-j}{-1-j} = \frac{2j}{2} = j.$$

Note that $\mathrm{Re}\,(z) = 0$; $\mathrm{Im}\,(z) = 1$, which is positive part of the vertical axis; $|1| = 1$; and $\arg(z) = \mathrm{atan2}\,(+1, 0) = +\pi/2$.

6.20. $$\frac{(j\sqrt{3}-1)^2}{(2j)^2} = \frac{1-2j\sqrt{3}-3}{-4} = \frac{-2-2j\sqrt{3}}{-4} = \frac{1}{2}+j\frac{\sqrt{3}}{2}$$

Note that $\mathrm{Re}\,(z) = 1/2$; $\mathrm{Im}\,(z) = \sqrt{3}/2$, which is in I quadrant; $|1/2 + j\sqrt{3}/2| = 1$ which is on unit circle; and $\arg(z) = \mathrm{atan2}\left(\sqrt{3}/2, 1/2\right) = \pi/3$.

6.21. $$\left(\frac{1+j}{\sqrt{2}}\right)^{1000}+\left(\frac{1-j}{\sqrt{2}}\right)^{1000} = \left(\frac{1+j}{\sqrt{2}}\right)^{2\times 500}+\left(\frac{1-j}{\sqrt{2}}\right)^{2\times 500} = \left(\frac{2j}{2}\right)^{500}+\left(-\frac{2j}{2}\right)^{500}$$
$$= 2\left(j^{4}\right)^{125} = 2$$

Note that $\mathrm{Re}\,(z) = 2$; $\mathrm{Im}\,(z) = 0$, which is positive part of the horizontal axis; $|2| = 2$; and $\arg(z) = \mathrm{atan2}\,(0, 2) = 0°$.

6.22. $$\frac{-41+63j}{50}-\frac{6j+1}{1-7j} = -\frac{41}{50}+\frac{63}{50}j+\frac{6j+1}{1-7j}\,\frac{1+7j}{1+7j} = -\frac{41}{50}+\frac{63}{50}j-\frac{6j-42+1+7j}{1+49}$$
$$= \frac{-41+63j+41-13j}{50} = \frac{63-13}{50}j = j$$

Note that $\mathrm{Re}\,(z) = 0$; $\mathrm{Im}\,(z) = 1$, which is positive part of the vertical axis; $|1| = 1$; and $\arg(z) = \mathrm{atan2}\,(+1, 0) = +\pi/2$.

6.23. With the same initial idea as in P. 6.17, take advantage of complex number rotation into real number position. Note that $\arg\left(1/2 + j\sqrt{3}/2\right) = \pi/3$; therefore, the third power is correct rotation and that $6\,066 = 3 \times 2\,022$:

$$\left(\frac{1+j\sqrt{3}}{2}\right)^{6\,066}+\left(\frac{1-j\sqrt{3}}{2}\right)^{6\,066}$$
$$= \left(\frac{(1+j\sqrt{3})^2(1+j\sqrt{3})}{2^3}\right)^{2\,022}+\left(\frac{(1-j\sqrt{3})^2(1-j\sqrt{3})}{2^3}\right)^{2\,022}$$
$$= \left(\frac{(-2+2j\sqrt{3})(1+j\sqrt{3})}{8}\right)^{2\,022}+\left(\frac{(-2-2j\sqrt{3})(1-j\sqrt{3})}{8}\right)^{2\,022}$$

$$= \left(\frac{-2(1-j\sqrt{3})(1+j\sqrt{3})}{8\,4}\right)^{2022} + \left(\frac{-2(1+j\sqrt{3})(1-j\sqrt{3})}{8\,4}\right)^{2022}$$

$$\{(a-b)(a+b) = a^2 - b^2\}$$

$$= \left(-\frac{1-(-3)}{4}\right)^{2022} + \left(-\frac{1-(-3)}{4}\right)^{2022} = (-1)^{2022} + (-1)^{2022} = 1 + 1 = 2.$$

Note that $\mathrm{Re}\,(z) = 2$; $\mathrm{Im}\,(z) = 0$, which is positive part of the horizontal axis; $|2| = 2$; and $\arg(z) = \mathrm{atan2}\,(0, +2) = 0°$.

6.24. With the same initial idea as in P. 6.17, take advantage of complex number rotation into real number position. Note that $\arg\left(\sqrt{3}/2 + j/2\right) = +\pi/6$; therefore, the sixth power is correct rotation. In addition, $1/(2j) = -j/2$, $\sqrt{12} = 2\sqrt{3}$, $108 = 6 \times 18$, $216 = 6 \times 36$, and $6 = 3 \times 2$. So,

$$\left(\frac{\sqrt{3}}{2} + \frac{j}{2}\right)^{108} - \left(\frac{\sqrt{12}}{4} + \frac{1}{2j}\right)^{216}$$

$$= \left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right)^{6\times 18} - \left(\frac{\sqrt{4\times 3}}{4} + \frac{1}{2j}\,\frac{j}{j}\right)^{6\times 36}$$

$$= \left(\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right)^3\right)^{2\times 18} - \left(\left(\frac{2\sqrt{3}}{4\,2} - j\frac{1}{2}\right)^3\right)^{2\times 36}$$

$$= \left(\frac{(\sqrt{3}+j)^3}{2^3}\right)^{2\times 18} - \left(\frac{(\sqrt{3}-j)^3}{2^3}\right)^{2\times 36}$$

$$\{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \quad \text{see Fig. 2.1}\}$$

$$= \left(\frac{3\sqrt{3} + 8\,9j - 3\sqrt{3} - j}{2^3}\right)^{2\times 18} - \left(\frac{3\sqrt{3} - 8\,9j - 3\sqrt{3} + j}{2^3}\right)^{2\times 36}$$

$$= \left(\frac{8j}{2^3}\right)^{2\times 18} - \left(\frac{-8j}{2^3}\right)^{2\times 36} = \left(\frac{8^2 j^2}{2^6}\right)^{18} - \left(\frac{(-1)^2\,8^2\,j^2}{2^6}\right)^{36} = (-1)^{18} - (-1)^{36}$$

$$= 0.$$

Note that $\mathrm{Re}\,(z) = 0 \mathit{and}\ \mathrm{Im}\,(z) = 0$, which is at the origin of the complex plane.

6.2 Polar Forms

6.25. There are multiple forms of complex numbers, all interchangeable.

Reminder: Polar form of complex numbers (see Figs. 5.1, 6.1, and 6.2):

$$z = a + jb = |z|\, e^{j\varphi}$$
$$|z|^2 = z\, z^* = a^2 + b^2 \quad \text{and,} \quad \varphi = \text{atan2}\left(\text{Im}\,(z)\,, \text{Re}\,(z)\right) = \text{atan2}\,(b, a)$$

$$z = j \;\; \therefore \;\; \text{Re}\,(z) = 0,\; \text{Im}\,(z) = 1$$
$$|z| = \sqrt{0^2 + 1^2} = 1$$
$$\arg(z) = \text{atan2}\,(+1, 0) = \frac{\pi}{2}$$
$$\therefore$$
$$z = 1\, e^{j\,\pi/2}.$$

6.26. Algebraic form of z is converted into its equivalent polar form as

$$z = 2 - j \;\; \therefore \;\; \text{Re}\,(z) = 2,\; \text{Im}\,(z) = -1$$
$$|z| = \sqrt{(2^2 + (-1)^2)} = \sqrt{5}$$
$$\arg(z) = \text{atan2}\,(-1, 2) \;\; \left(\text{in } IV \text{ quadrant}, \;\; \approx -26.56^\circ\right) = -\arctan\left(\frac{1}{2}\right)$$
$$\therefore$$
$$z = \sqrt{5}\, e^{-j\, \arctan(1/2)}.$$

6.27. Algebraic form of z is converted into its equivalent polar form as

$$z = (2 - j)^6 = (2 - j)^{2\times 3} = (4 - 4j - 1)^3 = (3 - 4j)^2\,(3 - 4j) = (9 - 24j - 16)(3 - 4j)$$
$$= (-7 - 24j)(3 - 4j) = -21 + 28j - 72j - 96 = -117 - 44j.$$

Therefore,

$$z = -117 - 44j \;\; \therefore \;\; \text{Re}\,(z) = -117,\; \text{Im}\,(z) = -44$$
$$|z| = \sqrt{(-117)^2 + (-44)^2} = 125$$
$$\arg(z) = \text{atan2}\,(-44, -117) \;\; \left(\text{in } III \text{ quadrant}, \;\; \approx -159.39^\circ\right) = \pm\pi + \arctan\left(\frac{44}{117}\right)$$
$$\therefore$$
$$z = 5^3\, e^{j\,\left(\pm\pi + \arctan(44/117)\right)}.$$

6.28.
$$z = j^{2020} = j^{4\times 505} = \left(j^4\right)^{505} = 1^{505} = 1$$
$$\text{therefore,} \quad |z| = 1, \quad \text{and} \quad \angle z = 2\pi = 0$$

6.29. Algebraic form of z is converted into its equivalent polar form as

$$z = \frac{(1+j)^{200}(6+2j) - (1-j)^{198}(3-j)}{(1+j)^{196}(23-7j) + (1-j)^{194}(10+2j)} = \frac{(1+j)^{2\times 100}\,2(3+j) - (1-j)^{2\times 99}(3-j)}{(1+j)^{2\times 98}(23-7j) + (1-j)^{2\times 97}\,2(5+j)}$$
$$= \frac{(2j)^{100}\,2(3+j) - (-2j)^{99}(3-j)}{(2j)^{98}(23-7j) + (-2j)^{97}\,2(5+j)} = \frac{2^{99}\times 4(3+j) - 2^{99}j(3-j)}{-2^{98}(23-7j) - 2^{98}j(5+j)}$$
$$= \frac{2^{99}(12+4j-3j-1)}{-2^{98}(23-7j+5j-1)} = \frac{2(11+j)}{-2(11-j)}\,\frac{11+j}{11+j} = \frac{(11+j)^2}{-(11-j)(11+j)}$$
$$= \frac{121+22j-1}{-(121+1)} = \frac{120+22j}{-122} = -\frac{60+11j}{61} = -\frac{60}{61} - j\,\frac{11}{61}.$$

Therefore,

$$z = \frac{11+j}{-(11-j)} = -\frac{60}{61} - j\,\frac{11}{61}$$

$\therefore$

$$|z| = \frac{|11+j|}{|-1||11-j|} = \frac{\sqrt{122}}{\sqrt{122}} = 1, \quad \text{(i.e., on unit circle)}$$

$$\arg(z) = \operatorname{atan2}(-11/61, -60/61) \quad \left(\text{in } III \text{ quadrant}, \quad \approx -169.61^\circ\right) = \pm\pi + \arctan\left(\frac{11}{60}\right)$$

$\therefore$

$$z = 1\, e^{j\left(\pm\pi + \arctan(11/60)\right)}.$$

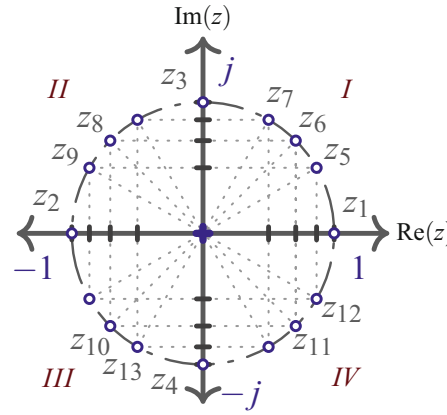
6.3 Complex Plane

Given algebraic form of a complex number, its module may be calculated simply by Pythagora's theorem and shown in unit circle (see Figs. 5.1 and 6.1), where horizontal axis is interpreted as Re (z) and vertical axis as Im (z).

6.30. Given $z = 1$, then (see Fig. 6.3)

$$z_1 = 1 \quad \therefore \quad \text{Re}\,(z) = 1,\ \text{Im}\,(z) = 0 \quad \therefore \quad |z| = \sqrt{(0)^2 + (1)^2} = 1$$
$$\arg(z) = \operatorname{atan2}(0, +1) \quad \left(\text{in positive part of horizontal axes}\right)$$
$$= \arctan\left(\frac{0}{1}\right) = 0.$$

Fig. 6.3 Example P.6.30



6.31. Given $z_2 = -1$, then (see Fig. 6.3)

$$z_2 = -1 \;\;\therefore\;\; \mathrm{Re}\,(z) = -1,\ \mathrm{Im}\,(z) = 0 \;\;\therefore\;\; |z| = \sqrt{(0)^2 + (-1)^2} = 1$$
$$\arg(z) = \mathrm{atan2}\,(0, -1) \quad \text{(in negative part of horizontal axes)}$$
$$= \pm\pi + \arctan\left(\frac{0}{1}\right) = \pm\pi.$$

6.32. Given $z_3 = j$, then (see Fig. 6.3)

$$z_3 = j \;\;\therefore\;\; \mathrm{Re}\,(z) = 0,\ \mathrm{Im}\,(z) = 1 \;\;\therefore\;\; |z| = \sqrt{(1)^2 + (0)^2} = 1$$
$$\arg(z) = \mathrm{atan2}\,(1, 0) \quad \text{(in positive part of vertical axes)}$$
$$= \arctan\left(\frac{1}{0}\right) = \arctan(\infty) = \frac{\pi}{2}.$$

6.33. Given $z_4 = -j$, then (see Fig. 6.3)

$$z_4 = -j \;\;\therefore\;\; \mathrm{Re}\,(z) = 0,\ \mathrm{Im}\,(z) = -1 \;\;\therefore\;\; |z| = \sqrt{(1)^2 + (0)^2} = 1$$
$$\arg(z) = \mathrm{atan2}\,(-1, 0) \quad \text{(in negative part of vertical axes)}$$
$$= \arctan\left(\frac{-1}{0}\right) = -\arctan(\infty) = -\frac{\pi}{2}.$$

6.34. Given $z_5 = \sqrt{3}/2 + j\,1/2$, then (see Fig. 6.3)

$$z_5 = \frac{\sqrt{3}}{2} + j\frac{1}{2} \;\;\therefore\;\; \mathrm{Re}\,(z) = \frac{\sqrt{3}}{2},\ \mathrm{Im}\,(z) = \frac{1}{2} \;\;\therefore\;\; |z| = \sqrt{\left(\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right)^2} = 1$$
$$\arg(z) = \mathrm{atan2}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \quad (\text{in } I \text{ quadrant}) = \arctan\left(\frac{1/2}{\sqrt{3}/2}\right) = \frac{\pi}{6}.$$

6.35. Given $z_6 = \sqrt{2}/2 + j\sqrt{2}/2$, then (see Fig. 6.3)

$$z_6 = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \;\;\therefore\;\; \mathrm{Re}\,(z) = \frac{\sqrt{2}}{2},\; \mathrm{Im}\,(z) = \frac{\sqrt{2}}{2} \;\;\therefore\;\; |z| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1$$

$$\arg(z) = \mathrm{atan2}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \;\;(\text{in } I \text{ quadrant}) = \arctan\left(\frac{\cancel{\sqrt{2}/2}}{\cancel{\sqrt{2}/2}}\,1\right) = \frac{\pi}{4}.$$

6.36. Given $z_7 = 1/2 + j\sqrt{3}/2$, then (see Fig. 6.3)

$$z_7 = \frac{1}{2} + j\frac{\sqrt{3}}{2} \;\;\therefore\;\; \mathrm{Re}\,(z) = \frac{1}{2},\; \mathrm{Im}\,(z) = \frac{\sqrt{3}}{2} \;\;\therefore\;\; |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\arg(z) = \mathrm{atan2}\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \;\;(\text{in } I \text{ quadrant}) = \arctan\left(\frac{\cancel{\sqrt{3}/2}}{\cancel{1/2}}\right) = \frac{\pi}{3}.$$

6.37. Given $z_8 = -\sqrt{2}/2 + j\sqrt{2}/2$, then (see Fig. 6.3)

$$z_8 = -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \;\;\therefore\;\; \mathrm{Re}\,(z) = -\frac{\sqrt{2}}{2},\; \mathrm{Im}\,(z) = \frac{\sqrt{2}}{2} \;\;\therefore\;\; |z| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2} = 1$$

$$\arg(z) = \mathrm{atan2}\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \;\;(\text{in } II \text{ quadrant})$$

$$= \pm\pi + \arctan\left(\frac{\cancel{\sqrt{2}/2}}{\cancel{-\sqrt{2}/2}\,-1}\right) = \pm\pi + \arctan(-1) = \frac{3\pi}{4}.$$

6.38. Given $z_9 = -\sqrt{3}/2 + j\,1/2$, then (see Fig. 6.3)

$$z_9 = -\frac{\sqrt{3}}{2} + j\frac{1}{2} \;\;\therefore\;\; \mathrm{Re}\,(z) = -\frac{\sqrt{3}}{2},\; \mathrm{Im}\,(z) = \frac{1}{2} \;\;\therefore\;\; |z| = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$$

$$\arg(z) = \mathrm{atan2}\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \;\;(\text{in } II \text{ quadrant})$$

$$= \pm\pi + \arctan\left(\frac{\cancel{1/2}}{\cancel{-\sqrt{3}/2}}\right) = \pm\pi + \arctan(-1/\sqrt{3}) = \frac{5\pi}{6}.$$

6.39. Given $z_{10} = -\sqrt{2}/2 - j\sqrt{2}/2$, then (see Fig. 6.3)

$$z_{10} = --\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \;\;\therefore\;\; \mathrm{Re}\,(z) = -\frac{\sqrt{2}}{2},\; \mathrm{Im}\,(z) = -\frac{\sqrt{2}}{2} \;\;\therefore\;\; |z| = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2} = 1$$

$$\arg(z) = \mathrm{atan2}\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \;\;(\text{in } III \text{ quadrant})$$

$$= \pm\pi + \arctan\left(\frac{\cancel{-\sqrt{2}/2}}{\cancel{-\sqrt{2}/2}}\,1\right) = \frac{5\pi}{4} \equiv -\frac{3\pi}{4}.$$

6.40. Given $z_{11} = \sqrt{2}/2 - j\sqrt{2}/2$, then (see Fig. 6.3)

$$z_{11} = \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \;\therefore\; \mathrm{Re}\,(z) = \frac{\sqrt{2}}{2},\ \mathrm{Im}\,(z) = -\frac{\sqrt{2}}{2} \;\therefore\; |z| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2} = 1$$

$$\arg(z) = \mathrm{atan2}\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \quad (\text{in } IV \text{ quadrant})$$

$$= \arctan\left(\frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1\right) = -\frac{\pi}{4}.$$

6.41. Given $z_{12} = \frac{\sqrt{3}}{2} - j\frac{1}{2}$, then (see Fig. 6.3)

$$z_{12} = \frac{\sqrt{3}}{2} - j\frac{1}{2} \;\therefore\; \mathrm{Re}\,(z) = \frac{\sqrt{3}}{2},\ \mathrm{Im}\,(z) = -\frac{1}{2} \;\therefore\; |z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = 1$$

$$\arg(z) = \mathrm{atan2}\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \quad (\text{in } IV \text{ quadrant})$$

$$= \arctan\left(\frac{-1/2}{\sqrt{3}/2}\right) = \arctan(-1/\sqrt{3}) = -\frac{\pi}{6}.$$

6.42. Given $z_{13} = -1/2 - j\sqrt{3}/2$, then (see Fig. 6.3)

$$z_{13} = -\frac{1}{2} - j\frac{\sqrt{3}}{2} \;\therefore\; \mathrm{Re}\,(z) = -\frac{1}{2},\ \mathrm{Im}\,(z) = -\frac{\sqrt{3}}{2} \;\therefore\; |z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\arg(z) = \mathrm{atan2}\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \quad (\text{in } III \text{ quadrant})$$

$$= \pm\pi + \arctan\left(\frac{-\sqrt{3}/2}{-1/2}\right) = \pm\pi + \arctan\left(\sqrt{3}\right) = \frac{4\pi}{3} \equiv -\frac{2\pi}{3}.$$

6.4 Euler Identity

Reminder: Euler's formula is yet another form of complex numbers, explicitly showing the relationships of its real and imaginary parts with the basic trigonometric functions, as

$$z = a + jb = |z|e^{j\varphi} = |z|\,(\cos\varphi + j\sin\varphi)\,.$$

6.43. Recall that (see Fig. 6.1) the real part of a complex number equals to the length of horizontal cathetus and its imaginary part equals to the length of the vertical cathetus.

Complex numbers in this example given in exponential form thus, by exploiting Euler's equation, phase and module of a complex number are found by inspection:

$$z = |z|e^{j\varphi} = e^{j\,0} \quad \therefore \quad |z| = 1, \quad \arg(z) = \varphi = 0$$
$$z = |z|\,(\cos\varphi + j\,\sin\varphi) = \ 1\ (\cos 0 + j\,\sin 0)$$
which is identical to
$$= 1 + j\,0 = 1 \quad \text{(see A. 6.30)}$$

Complex conjugate number is simply with changed sign of the imaginary part, i.e., if

$$z = a + jb, \quad \text{then} \quad z^* = a - jb; \quad \text{then,}$$
$$z\,z^* = (a + jb)(a - jb) = a^2 - \cancel{jab} + \cancel{jab} - j^2b^2 = a^2 + b^2 = |z|^2 = c^2, \ \text{Pythagora's theorem.}$$

Therefore, in Euler's form

$$z = \cos 0 + j\,\sin 0 \quad \therefore \quad z^* = \cos 0 - j\,\sin 0 = 1$$
or in polar form
$$z = e^{j\,0} \quad \therefore \quad z^* = e^{-j\,0} = 1 \ \text{ in both cases;}$$

in other words, complex conjugate of a real number is the same number (naturally because to say "real number" is to say Im $(z) = 0$). Also, product of $z\,z^* = |z|^2$, which is verified as

$$z\,z^* = e^{j\,0}\,e^{-j\,0} = e^{(j\,0)+(-j\,0)} = e^0 = 1 = |1|^2 = |z|^2 \quad \text{or}$$
$$z\,z^* = (\cos 0 - j\,\sin 0)\,(\cos 0 - j\,\sin 0) = 1 \times 1 = 1^2 = |z|^2.$$

6.44. By inspection,

$$z = |z|e^{j\varphi} = e^{j\,\frac{\pi}{6}} \quad \therefore \quad |z| = 1, \quad \arg(z) = \varphi = \frac{\pi}{6}$$
$$z = |z|\,(\cos\varphi + j\,\sin\varphi) = \ 1\ \left(\cos\frac{\pi}{6} + j\,\sin\frac{\pi}{6}\right)$$
which is identical to
$$= \frac{\sqrt{3}}{2} + j\,\frac{1}{2} \quad \text{(see P. 6.34).}$$

Therefore, in Euler's form

$$z = \cos\frac{\pi}{6} + j\,\sin\frac{\pi}{6} \quad \therefore \quad z^* = \cos\frac{\pi}{6} - j\,\sin\frac{\pi}{6}$$
or in polar form
$$z = e^{j\,\frac{\pi}{6}} \quad \therefore \quad z^* = e^{-j\,\frac{\pi}{6}}.$$

Also, product of $z\,z^* = |z|^2$ is therefore

$$z\,z^* = e^{j\frac{\pi}{6}}\,e^{-j\frac{\pi}{6}} = e^{(j\frac{\pi}{6})+(-j\frac{\pi}{6})} = e^0 = 1 = |1|^2 = |z|^2 \quad \text{or}$$

$$z\,z^* = \left(\cos\frac{\pi}{6} + j\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{6} - j\sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)$$

$$= \frac{3}{4} - j\frac{\sqrt{3}}{4} + j\frac{\sqrt{3}}{4} - j^2\frac{1}{4} = \frac{3}{4} + \frac{1}{4} = 1 = 1^2 = |z|^2.$$

6.45. By inspection,

$$z = |z|e^{j\varphi} = e^{j\frac{\pi}{4}} \quad \therefore \quad |z| = 1, \quad \arg(z) = \varphi = \frac{\pi}{4}$$

$$z = |z|\,(\cos\varphi + j\sin\varphi) = 1\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)$$

which is identical to

$$= \frac{\sqrt{2}}{2} + j\,\frac{\sqrt{2}}{2} \quad \text{(see A. 6.35)}.$$

Therefore, in Euler's form

$$z = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4} \quad \therefore \quad z^* = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4}$$

or in polar form

$$z = e^{j\frac{\pi}{4}} \quad \therefore \quad z^* = e^{-j\frac{\pi}{4}}.$$

Also, product of $z\,z^* = |z|^2$ is therefore

$$z\,z^* = e^{j\frac{\pi}{4}}\,e^{-j\frac{\pi}{4}} = e^{(j\frac{\pi}{4})+(-j\frac{\pi}{4})} = e^0 = 1 = |1|^2 = |z|^2 \quad \text{or}$$

$$z\,z^* = \left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right)$$

$$= \frac{1}{2} - j\frac{\sqrt{2}}{4} + j\frac{\sqrt{2}}{4} - j^2\frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1 = 1^2 = |z|^2.$$

6.46. By inspection,

$$z = |z|e^{j\varphi} = e^{j\frac{\pi}{3}} \quad \therefore \quad |z| = 1, \quad \arg(z) = \varphi = \frac{\pi}{3}$$

$$z = |z|\,(\cos\varphi + j\sin\varphi) = 1\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right)$$

which is identical to

$$= \frac{1}{2} + j\,\frac{\sqrt{3}}{2} \quad \text{(see A. 6.36)}.$$

Therefore, in Euler's form

$$z = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} \quad \therefore \quad z^* = \cos\frac{\pi}{3} - j\sin\frac{\pi}{3}$$

or in polar form

$$z = e^{j\frac{\pi}{3}} \quad \therefore \quad z^* = e^{-j\frac{\pi}{3}}.$$

Also, product of $z\,z^* = |z|^2$ is therefore

$$z\,z^* = e^{j\frac{\pi}{3}}\,e^{-j\frac{\pi}{3}} = e^{(j\frac{\pi}{3})+(-j\frac{\pi}{3})} = e^0 = 1 = |1|^2 = |z|^2 \quad \text{or}$$

$$z\,z^* = \left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3} - j\sin\frac{\pi}{3}\right) = \left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{4} - j\frac{\sqrt{3}}{4} + j\frac{\sqrt{3}}{4} - j^2\frac{3}{4} = \frac{1}{4} + \frac{3}{4} = 1 = 1^2 = |z|^2.$$

6.47. By inspection,

$$z = |z|e^{j\varphi} = e^{j\frac{\pi}{2}} \quad \therefore \quad |z| = 1, \quad \arg(z) = \varphi = \frac{\pi}{2}$$

$$z = |z|\,(\cos\varphi + j\sin\varphi) = 1\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right)$$

which is identical to

$$= 0 + j \quad \text{(see A. 6.32)}.$$

Therefore, in Euler's form

$$z = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} \quad \therefore \quad z^* = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$

or in polar form

$$z = e^{j\frac{\pi}{2}} \quad \therefore \quad z^* = e^{-j\frac{\pi}{2}}.$$

Also, product of $z\,z^* = |z|^2$ is therefore

$$z\,z^* = e^{j\frac{\pi}{2}}\,e^{-j\frac{\pi}{2}} = e^{(j\frac{\pi}{2})+(-j\frac{\pi}{2})} = e^0 = 1 = |1|^2 = |z|^2 \quad \text{or}$$

$$z\,z^* = \left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right) = (0 + j\,1)\;(0 - j\,1)$$

$$= -j^2\,1^2 = 1 = 1^2 = |z|^2.$$

6.48. By inspection,

$$z = |z|e^{j\varphi} = e^{j\frac{3\pi}{4}} \quad \therefore \quad |z| = 1, \quad \arg(z) = \varphi = \frac{3\pi}{4}$$

$$z = |z|\,(\cos\varphi + j\sin\varphi) = 1\left(\cos\frac{3\pi}{4} + j\sin\frac{3\pi}{4}\right)$$

which is identical to

$$= -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \quad \text{(see A. 6.37)}.$$

Therefore, in Euler's form

$$z = \cos\frac{3\pi}{4} + j\sin\frac{3\pi}{4} \quad \therefore \quad z^* = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}$$

or in polar form

$$z = e^{j\frac{3\pi}{4}} \quad \therefore \quad z^* = e^{-j\frac{3\pi}{4}}.$$

Also, product of $z\,z^* = |z|^2$ is therefore

$$z\,z^* = e^{j\frac{3\pi}{4}}\,e^{-j\frac{3\pi}{4}} = e^{(j\frac{3\pi}{4})+(-j\frac{3\pi}{4})} = e^0 = 1 = |1|^2 = |z|^2 \quad \text{or}$$

$$z\,z^* = \left(\cos\frac{3\pi}{4} + j\sin\frac{3\pi}{4}\right)\left(\cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2} + j\,\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2} - j\,\frac{\sqrt{2}}{2}\right)$$

$$= \frac{1}{2} + j\cancel{\frac{1}{2}} - j\cancel{\frac{1}{2}} - j^2\,\frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1 = 1^2 = |z|^2.$$

6.49. By inspection,

$$z = |z|e^{j\varphi} = e^{j\frac{5\pi}{6}} \quad \therefore \quad |z| = 1, \quad \arg(z) = \varphi = \frac{5\pi}{6}$$

$$z = |z|\,(\cos\varphi + j\sin\varphi) = 1\left(\cos\frac{5\pi}{6} + j\sin\frac{5\pi}{6}\right)$$

which is identical to

$$= -\frac{\sqrt{3}}{2} + j\frac{1}{2} \quad \text{(see P. 6.38)}.$$

Therefore, in Euler's form

$$z = \cos\frac{5\pi}{6} + j\sin\frac{5\pi}{6} \quad \therefore \quad z^* = \cos\frac{5\pi}{6} - j\sin\frac{5\pi}{6}$$

or in polar form

$$z = e^{j\frac{5\pi}{6}} \quad \therefore \quad z^* = e^{-j\frac{5\pi}{6}}.$$

Also, product of $z\,z^* = |z|^2$ is therefore

$$z\,z^* = e^{j\frac{5\pi}{6}}\,e^{-j\frac{5\pi}{6}} = e^{(j\frac{5\pi}{6})+(-j\frac{5\pi}{6})} = e^0 = 1 = |1|^2 = |z|^2 \quad \text{or}$$

$$z\,z^* = \left(\cos\frac{5\pi}{6} + j\sin\frac{5\pi}{6}\right)\left(\cos\frac{5\pi}{6} - j\sin\frac{5\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2} + j\,\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2} - j\,\frac{1}{2}\right)$$

$$= \frac{3}{4} + j\cancel{\frac{\sqrt{3}}{4}} - j\cancel{\frac{\sqrt{3}}{4}} - j^2\,\frac{1}{4} = \frac{3}{4} + \frac{1}{4} = 1 = 1^2 = |z|^2.$$

6.50. By inspection,

$$z = |z|e^{j\varphi} = e^{j\,\pi} \quad \therefore \quad |z| = 1, \quad \arg(z) = \varphi = \pi$$
$$z = |z|\,(\cos\varphi + j\sin\varphi) = \ 1\ (\cos\pi + j\sin\pi)$$
which is identical to
$$= -1 + j\,0 \quad \text{(see A. 6.32)}.$$

Therefore, in Euler's form

$$z = \cos\pi + j\sin\pi \quad \therefore \quad z^* = \cos\pi - j\sin\pi$$
or in polar form
$$z = e^{j\,\pi} \quad \therefore \quad z^* = e^{-j\,v}.$$

Also, product of $z\,z^* = |z|^2$ is therefore

$$z\,z^* = e^{j\,\pi}\,e^{-j\,\pi} = e^{(j\,\pi)+(-j\,\pi)} = e^0 = 1 = |1|^2 = |z|^2 \quad \text{or}$$
$$z\,z^* = (\cos\pi + j\sin\pi)\ (\cos\pi - j\sin\pi) = (-1 + j\,0)\ (-1 - j\,0) = 1 = 1^2 = |z|^2.$$

6.51. By inspection,

$$z = |z|e^{j\varphi} = e^{j\,\frac{5\pi}{4}} \quad \therefore \quad |z| = 1, \quad \arg(z) = \varphi = \frac{5\pi}{4} \equiv -\frac{3\pi}{4}$$
$$z = |z|\,(\cos\varphi + j\sin\varphi) = \ 1\ \left(\cos\frac{5\pi}{4} + j\sin\frac{5\pi}{4}\right)$$
which is identical to
$$= -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \quad \text{(see A. 6.39)}.$$

Therefore, in Euler's form

$$z = \cos\frac{5\pi}{4} + j\sin\frac{5\pi}{4} \quad \therefore \quad z^* = \cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4}$$
or in polar form
$$z = e^{j\,\frac{5\pi}{4}} \quad \therefore \quad z^* = e^{-j\,\frac{5\pi}{4}}.$$

Also, product of $z\,z^* = |z|^2$ is therefore

$$z\,z^* = e^{j\,\frac{5\pi}{4}}\,e^{-j\,\frac{5\pi}{4}} = e^{(j\,\frac{5\pi}{4})+(-j\,\frac{5\pi}{4})} = e^0 = 1 = |1|^2 = |z|^2 \quad \text{or}$$
$$z\,z^* = \left(\cos\frac{5\pi}{4} + j\sin\frac{5\pi}{4}\right)\left(\cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2} - j\,\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2} + j\,\frac{\sqrt{2}}{2}\right)$$

$$= \frac{1}{2} - j\cancel{\frac{1}{2}} + j\cancel{\frac{1}{4}} - j^2\,\frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1 = 1^2 = |z|^2.$$

6.52. By inspection,

$$z = |z|e^{j\varphi} = e^{j\,\frac{3\pi}{2}} \quad \therefore \quad |z| = 1, \quad \arg(z) = \varphi = \frac{3\pi}{2} \equiv -\frac{\pi}{2}$$

$$z = |z|\,(\cos\varphi + j\sin\varphi) = 1\left(\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2}\right)$$

which is identical to

$= 0 - j\,1$ (see A. 6.33).

Therefore, in Euler's form

$$z = \cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2} \quad \therefore \quad z^* = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$

or in polar form

$$z = e^{j\,\frac{3\pi}{2}} \quad \therefore \quad z^* = e^{-j\,\frac{3\pi}{2}}.$$

Also, product of $z\,z^* = |z|^2$ is therefore

$$z\,z^* = e^{j\,\frac{3\pi}{2}}\,e^{-j\,\frac{3\pi}{2}} = e^{(j\,\frac{3\pi}{2})+(-j\,\frac{3\pi}{2})} = e^0 = 1 = |1|^2 = |z|^2 \quad \text{or}$$

$$z\,z^* = \left(\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2}\right)\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) = (0 - j\,1)\;(0 + j\,1)$$

$$= -j^2\,1 = 1 = 1^2 = |z|^2.$$

6.53. By inspection,

$$z = |z|e^{j\varphi} = e^{j\,\frac{7\pi}{4}} \quad \therefore \quad |z| = 1, \quad \arg(z) = \varphi = \frac{7\pi}{4} \equiv -\frac{\pi}{4}$$

$$z = |z|\,(\cos\varphi + j\sin\varphi) = 1\left(\cos\frac{7\pi}{4} + j\sin\frac{7\pi}{4}\right)$$

which is identical to

$$= \frac{\sqrt{2}}{2} - j\,\frac{\sqrt{2}}{2} \quad \text{(see A. 6.40)}.$$

Therefore, in Euler's form

$$z = \cos\frac{7\pi}{4} + j\sin\frac{7\pi}{4} \quad \therefore \quad z^* = \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4}$$

or in polar form

$$z = e^{j\frac{7\pi}{4}} \quad \therefore \quad z^* = e^{-j\frac{7\pi}{4}}.$$

Also, product of $z\,z^* = |z|^2$ is therefore

$$z\,z^* = e^{j\frac{7\pi}{4}}\,e^{-j\frac{7\pi}{4}} = e^{(j\frac{7\pi}{4})+(-j\frac{7\pi}{4})} = e^0 = 1 = |1|^2 = |z|^2 \quad \text{or}$$

$$z\,z^* = \left(\cos\frac{7\pi}{4} + j\sin\frac{7\pi}{4}\right)\left(\cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4}\right) = \left(\frac{\sqrt{2}}{2} - j\,\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2} + j\,\frac{\sqrt{2}}{2}\right)$$

$$= \frac{1}{2} + j\frac{1}{2} - j\frac{1}{2} - j^2\frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1 = 1^2 = |z|^2.$$

6.54. By inspection,

$$z = |z|e^{j\varphi} = e^{j\frac{11\pi}{6}} \quad \therefore \quad |z| = 1, \quad \arg(z) = \varphi = \frac{11\pi}{6} \equiv -\frac{\pi}{6}$$

$$z = |z|\,(\cos\varphi + j\sin\varphi) = 1\left(\cos\frac{11\pi}{6} + j\sin\frac{11\pi}{6}\right)$$

which is identical to

$$= \frac{\sqrt{3}}{2} - j\,\frac{1}{2} \quad \text{(see A. 6.41)}.$$

Therefore, in Euler's form

$$z = \cos\frac{11\pi}{6} + j\sin\frac{11\pi}{6} \quad \therefore \quad z^* = \cos\frac{11\pi}{6} - j\sin\frac{11\pi}{6}$$

or in polar form

$$z = e^{j\frac{11\pi}{6}} \quad \therefore \quad z^* = e^{-j\frac{11\pi}{6}}.$$

Also, product of $z\,z^* = |z|^2$ is therefore

$$z\,z^* = e^{j\frac{11\pi}{6}}\,e^{-j\frac{11\pi}{6}} = e^{(j\frac{11\pi}{6})+(-j\frac{11\pi}{6})} = e^0 = 1 = |1|^2 = |z|^2 \quad \text{or}$$

$$z\,z^* = \left(\cos\frac{11\pi}{6} + j\sin\frac{11\pi}{6}\right)\left(\cos\frac{11\pi}{6} - j\sin\frac{11\pi}{6}\right) = \left(\frac{\sqrt{3}}{2} - j\,\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2} + j\,\frac{1}{2}\right)$$

$$= \frac{3}{4} - j\frac{\sqrt{3}}{4} + j\frac{\sqrt{3}}{4} - j^2\frac{1}{4} = \frac{3}{4} + \frac{1}{4} = 1 = 1^2 = |z|^2.$$

6.55. Given polar form, note that all numbers are positive and that absolute value of a positive number is the same number; thus, by definition

$$z = \frac{1}{2}\,e^{j\frac{\pi}{3}} \quad \therefore \quad |z| = \frac{1}{2} \quad \text{and} \quad \arg(z) = \varphi = \frac{\pi}{3}.$$

6.56. Given polar form, negative number must be also counted as complex, i.e.,

$$z = -2\,e^{-j\frac{5\pi}{6}} = -1 \times 2 \times e^{-j\frac{5\pi}{6}} = e^{j\pi} \times 2 \times e^{-j\frac{5\pi}{6}} = 2\,e^{j\left(\pi-\frac{5\pi}{6}\right)} = 2\,e^{j\frac{\pi}{6}}$$

$$|z| = 2 \quad \text{and} \quad \arg(z) = \varphi = \frac{\pi}{6}.$$

6.57. Given polar form, negative number must be also counted as complex, i.e.,

$$z = -\frac{1}{2}\,e^{j\frac{7\pi}{4}} = -1 \times \frac{1}{2} \times e^{j\frac{7\pi}{4}} = e^{j\pi} \times \frac{1}{2} \times e^{j\frac{7\pi}{4}} = \frac{1}{2}\,e^{j\left(\pi+\frac{7\pi}{4}\right)} = \frac{1}{2}\,e^{j\frac{11\pi}{4}}$$

$$= \frac{1}{2}\,e^{j\left(\frac{8\pi}{4}+\frac{3\pi}{4}\right)} = \left\{\frac{8\pi}{4} = 2\pi = 0\right\} = \frac{1}{2}\,e^{j\frac{3\pi}{4}}$$

$$|z| = \frac{1}{2} \quad \text{and} \quad \arg(z) = \varphi = \frac{3\pi}{4}.$$

6.58. Given polar form, by definition

$$z = \left(\sqrt{2}\,e^{j\frac{\pi}{4}}\right)^2 = \left(\sqrt{2}\right)^2 \left(e^{j\frac{\pi}{4}}\right)^2 = 2\,e^{j\frac{\pi}{4}\times 2} = 2\,e^{j\frac{\pi}{2}}$$

or

$$= 2\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right) = 2\,(0 + j\,1) = 2j;$$

therefore,

$$|z| = 2 \quad \text{and} \quad \arg(z) = \varphi = \frac{\pi}{2}.$$

6.59. Given polar form,

$$z = \left(3\,e^{j\frac{\pi}{3}}\right)^3 = 3^3\,e^{j\frac{\pi}{3}\times 3} = 27\,e^{j\pi} \quad \text{or} \quad = 27\,(-1) = -27;$$

therefore,

$$|z| = |-27| = 27 \quad \text{and} \quad \arg(z) = \varphi = \pi.$$

6.60. Given polar form,

$$z = \left(4\,e^{j\frac{5\pi}{6}}\right)^5 = 4^5\,e^{j\frac{25\pi}{6}} = 4^5\,e^{j\frac{2\pi\times 24+\pi}{6}} = \{2\pi \times 24 = 0\} = 2^{10}\,e^{j\frac{\pi}{6}} = 1024\,e^{j\frac{\pi}{6}};$$

therefore,

$$|z| = 2^{10} = 1024 \quad \text{and} \quad \arg(z) = \varphi = \frac{\pi}{6}.$$

6.61. Given polar form,

$$z = \left(0.2\,e^{j\frac{7\pi}{4}}\right)^4 = \left(\frac{1}{5}\right)^4 e^{j\frac{7\pi}{4}\times 4} = \frac{1}{625}\,e^{j2\pi\times 3+\pi} = \{2\pi \times 3 = 0\} = \frac{1}{625}\,e^{j\pi}$$

$$\text{or } = \frac{1}{625}(-1) = -\frac{1}{625};$$

therefore,

$$|z| = \left|-\frac{1}{625}\right| = \frac{1}{625} \quad \text{and} \quad \arg(z) = \varphi = \pi.$$

6.62. Given polar form, negative number must be also counted as complex, i.e.,

$$z = \left(-3\,e^{-j\frac{\pi}{6}}\right)^3 = (-3)^3\,e^{-j\frac{\pi}{6}\times 3} = -27\,e^{-j\frac{\pi}{2}} = -1\times 27\times e^{-j\frac{\pi}{2}} = e^{j\pi}\times 27\times e^{-j\frac{\pi}{2}}$$
$$= 27\,e^{j\left(\pi-\frac{\pi}{2}\right)} = 27\,e^{j\frac{\pi}{2}} \quad \text{or} \quad = 27j;$$

therefore,

$$|z| = 27 \quad \text{and} \quad \arg(z) = \varphi = \frac{\pi}{2}.$$

6.5 Rational Powers

6.63. Radicals are converted into fractional powers, so that

$$z_1 = \sqrt{j} = j^{\frac{1}{2}} = \left(e^{j\frac{\pi}{2}}\right)^{\frac{1}{2}} = e^{j\frac{\pi}{4}} \quad \text{or} \quad = \left(\cos\frac{\pi}{4} + j\,\sin\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + j\,\frac{\sqrt{2}}{2};$$

therefore,

$$\text{Re}\,(z) = \frac{\sqrt{2}}{2}, \quad \text{Im}\,(z) = \frac{\sqrt{2}}{2}, \quad |z| = 1 \quad \text{and} \quad \arg(z) = \varphi = \frac{\pi}{4}.$$

6.64. Radicals are converted into fractional powers, so that

$$z_2 = \sqrt{-j} = (-j)^{\frac{1}{2}} = \left(e^{-j\frac{\pi}{2}}\right)^{\frac{1}{2}} = e^{-j\frac{\pi}{4}} \quad \text{or} \quad = \cos\left(-\frac{\pi}{4}\right) + j\,\sin\left(-\frac{\pi}{4}\right)$$
$$\left\{\cos(-x) = \cos x, \quad \sin(-x) = -\sin x\right\}$$

$$= \cos\frac{\pi}{4} - j\,\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} - j\,\frac{\sqrt{2}}{2};$$

therefore,

$$\text{Re}\,(z) = \frac{\sqrt{2}}{2}, \quad \text{Im}\,(z) = -\frac{\sqrt{2}}{2}, \quad |z| = 1 \quad \text{and} \quad \arg(z) = \varphi = -\frac{\pi}{4}.$$

Note that $-j$ is complex conjugate of j.

6.65. Radicals are converted into fractional powers, so that

$$z_3 = \sqrt[3]{j} = \left(e^{j\frac{\pi}{2}}\right)^{\frac{1}{3}} = e^{j\frac{\pi}{6}} \quad \text{or} \quad = \left(\cos\frac{\pi}{6} + j\ \sin\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + j\,\frac{1}{2};$$

therefore,

$$\text{Re}\,(z) = \frac{\sqrt{3}}{2}, \quad \text{Im}\,(z) = \frac{1}{2}, \quad |z| = 1 \quad \text{and} \quad \arg(z) = \varphi = \frac{\pi}{6}.$$

6.66. Radicals are converted into fractional powers, so that

$$z_4 = \sqrt[4]{j} = \left(e^{j\frac{\pi}{2}}\right)^{\frac{1}{4}} = e^{j\frac{\pi}{8}} \quad \text{or} \quad = \left(\cos\frac{\pi}{8} + j\ \sin\frac{\pi}{8}\right);$$

therefore,

$$\text{Re}\,(z) = \cos\frac{\pi}{8}, \quad \text{Im}\,(z) = \sin\frac{\pi}{8}, \quad |z| = 1 \quad \text{and} \quad \arg(z) = \varphi = \frac{\pi}{8}.$$

6.67. Radicals are converted into fractional powers, so that

$$z_5 = \sqrt[5]{j} = \left(e^{j\frac{\pi}{2}}\right)^{\frac{1}{5}} = e^{j\frac{\pi}{10}} \quad \text{or} \quad = \left(\cos\frac{\pi}{10} + j\ \sin\frac{\pi}{10}\right)$$

; therefore,

$$\text{Re}\,(z) = \cos\frac{\pi}{10}, \quad \text{Im}\,(z) = \sin\frac{\pi}{10}, \quad |z| = 1 \quad \text{and} \quad \arg(z) = \varphi = \frac{\pi}{10}.$$

6.68. Radicals are converted into fractional powers, so that

$$z_6 = \sqrt{\frac{1}{2j}\ \frac{j}{j}} = \left(-\frac{1}{2}\,j\right)^{\frac{1}{2}} = \left(e^{j\pi} \times \frac{1}{2} \times e^{j\frac{\pi}{2}}\right)^{\frac{1}{2}} = \left(\frac{1}{2} \times e^{j\left(\pi+\frac{\pi}{2}\right)}\right)^{\frac{1}{2}} = \left(\frac{1}{2} \times e^{j\,\frac{3\pi}{2}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{2} \times e^{-j\,\frac{\pi}{2}}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\,e^{-j\,\frac{\pi}{2}\times\frac{1}{2}} = \frac{1}{\sqrt{2}}\,e^{-j\,\frac{\pi}{4}} = \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{\pi}{4}\right) + j\sin\left(-\frac{\pi}{4}\right)\right)$$

$\left\{\cos(-x) = \cos x, \quad \sin(-x) = -\sin x\right\}$

$$= \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right) = \frac{1}{2} - j\frac{1}{2};$$

therefore,

$$\text{Re}\,(z) = \frac{1}{2}, \quad \text{Im}\,(z) = -\frac{1}{2}, \quad |z| = \frac{\sqrt{2}}{2} \quad \text{and} \quad \arg(z) = \varphi = -\frac{\pi}{4}.$$

Note how the negative sign propagated and modified $\arg(z)$, as well as the equivalence between $3\pi/2 \equiv -\pi/2$, which is negative argument.

6.69. First,

$$1-j \;\;\therefore\;\; \mathrm{Re}\,(z)=1,\;\mathrm{Im}\,(z)=-1 \;\;\therefore\;\; |1-j|=\sqrt{1^2+(-1)^2}=\sqrt{2}$$

$$\therefore\;\; \arg(1-j)=\mathrm{atan2}\,(\mathrm{Im}\,(z)\,,\mathrm{Re}\,(z))=\mathrm{atan2}\,(-1,1)=-\frac{\pi}{4}$$

;therefore, its polar form is

$$1-j=\sqrt{2}\,e^{-j\frac{\pi}{4}}$$

so that

$$z_7=\sqrt{(1-j)^3}=\sqrt{\left(\sqrt{2}\,e^{-j\frac{\pi}{4}}\right)^3}=\left((\sqrt{2})^3\,e^{-j\frac{3\pi}{4}}\right)^{\frac{1}{2}}=2^{\frac{3}{2}\times\frac{1}{2}}\,e^{-j\frac{3\pi}{4}\times\frac{1}{2}}$$

$$=2^{\frac{3}{4}}\,e^{-j\frac{3\pi}{8}}=\sqrt[4]{8}\,e^{-j\frac{3\pi}{8}}\;\;\text{; therefore,}$$

$$\left|\sqrt{(1-j)^3}\right|=\sqrt[4]{8},\;\;\arg\left(\sqrt{(1-j)^3}\right)=-\frac{3\pi}{8}$$

$$\mathrm{Re}\left(\sqrt{(1-j)^3}\right)=\sqrt[4]{8}\,\cos\left(-j\frac{3\pi}{8}\right),\;\;\mathrm{Im}\left(\sqrt{(1-j)^3}\right)=\sqrt[4]{8}\,\sin\left(-j\frac{3\pi}{8}\right).$$

6.70. First,

$$\frac{1}{(1+j)^2}=\frac{1}{\cancel{1}+2j-\cancel{1}}=-\frac{1}{2}\,j=e^{j\pi}\,\frac{1}{2}\,e^{j\frac{\pi}{2}}=\frac{1}{2}\,e^{j\pi+\frac{\pi}{2}}=\frac{1}{2}\,e^{j\frac{3\pi}{2}}=\frac{1}{2}\,e^{-j\frac{\pi}{2}}$$

so that

$$z_8=\sqrt[3]{\frac{1}{(1+j)^2}}=\left(\frac{1}{2}\,e^{-j\frac{\pi}{2}}\right)^{\frac{1}{3}}=\frac{1}{\sqrt[3]{2}}\,e^{-j\frac{\pi}{2}\times\frac{1}{3}}=\frac{1}{\sqrt[3]{2}}\,e^{-j\frac{\pi}{6}}=\;\;\text{therefore}$$

$$\left|\sqrt[3]{\frac{1}{(1+j)^2}}\right|=\frac{1}{\sqrt[3]{2}},\;\;\arg\left(\sqrt[3]{\frac{1}{(1+j)^2}}\right)=-\frac{\pi}{6}$$

$$\mathrm{Re}\left(\sqrt[3]{\frac{1}{(1+j)^2}}\right)=\frac{1}{\sqrt[3]{2}}\,\cos\left(-\frac{\pi}{6}\right)=-\frac{1}{\sqrt[3]{2}}\,\frac{\sqrt{3}}{2}$$

$$\mathrm{Im}\left(\sqrt[3]{\frac{1}{(1+j)^2}}\right)=-\frac{1}{2}\,\frac{1}{\sqrt[3]{2}}.$$

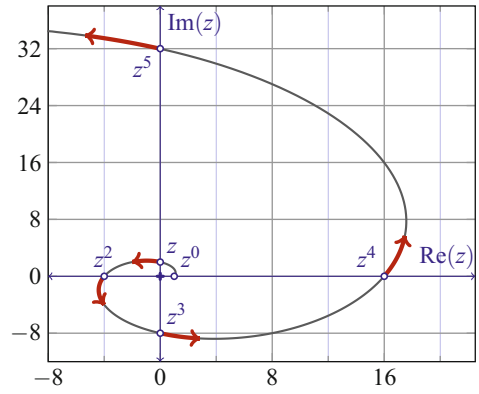
6.71. As powers on complex number increase, both its modulus and its argument increase as

$$\underline{n=0:}\;\;(2j)^0=1,\;\;\therefore\;\;\mathrm{Re}\,(z)=1,\;\;\mathrm{Im}\,(z)=0$$

$$\underline{n=1:}\;\;(2j)^1=2j,\;\;\therefore\;\;\mathrm{Re}\,(z)=0,\;\;\mathrm{Im}\,(z)=2$$

$$\underline{n=2:}\;\;(2j)^2=4j^2=-4,\;\;\therefore\;\;\mathrm{Re}\,(z)=-4,\;\;\mathrm{Im}\,(z)=0$$

Fig. 6.4 Example P.6.71



$$\underline{n=3:}\quad (2j)^3 = 8j^3 = -8j,\ \ \therefore\ \ \mathrm{Re}\,(z) = -0,\ \ \mathrm{Im}\,(z) = -8$$

$$\underline{n=4:}\quad (2j)^4 = 16j^4 = 16,\ \ \therefore\ \ \mathrm{Re}\,(z) = 16,\ \ \mathrm{Im}\,(z) = 0$$

$$\underline{n=5:}\quad (2j)^5 = 32j^5 = 32j,\ \ \therefore\ \ \mathrm{Re}\,(z) = 0,\ \ \mathrm{Im}\,(z) = 32$$

$$\vdots,$$

thus, as $n \to \infty$, creating progressively expanding spiralling trajectory where $|z| \to \infty$ and $\arg(z) \to \infty \times 2\pi$ (see Fig. 6.4).

6.6 Complex Equations

Due to symmetry, solutions of complex equations are symmetrically distributed in the complex plane.

6.72. Note that being the only two solutions of quadratic equation, z_1 and z_2 are located diametrically opposite in the complex plane, i.e., in I and III quadrants (Fig. 6.5):

$$z^2 = j \ \ \therefore\ \ \sqrt{z^2} = \sqrt{j} = j^{\frac{1}{2}}$$
$$|z| = j^{\frac{1}{2}} = \left(e^{j\frac{\pi}{2}}\right)^{\frac{1}{2}} = e^{j\frac{\pi}{4}}$$
$$\therefore$$

$$z_1 = \cos\frac{\pi}{4} + j\ \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$
$$z_2 = -\cos\frac{\pi}{4} - j\ \sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}$$

6.73. Note that being the only two solutions of quadratic equation, z_1 and z_2 are located diametrically opposite in the complex plane, i.e., in IV and II quadrants, Fig. 6.6:

$$z^2 = -j \ \ \therefore\ \ \sqrt{z^2} = \sqrt{-j}$$

$$|z| = (-j)^{\frac{1}{2}} = \left(e^{-j\frac{\pi}{2}}\right)^{\frac{1}{2}} = e^{-j\frac{\pi}{4}}$$

Fig. 6.5 Example P.6.72

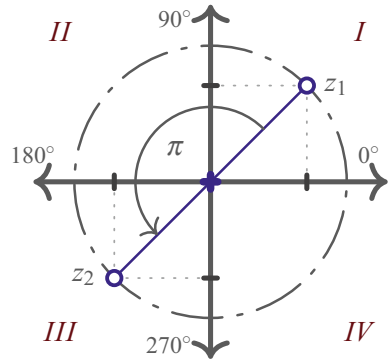
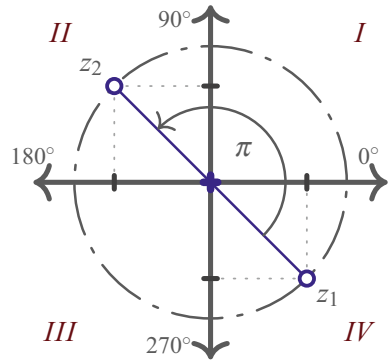


Fig. 6.6 Example P.6.73



$$\therefore$$

$$z_1 = \cos\frac{\pi}{4} - j\ \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}$$

$$z_2 = -\cos\frac{\pi}{4} + j\ \sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}.$$

6.74. By separating real and complex parts of this complex equation, it follows that

$$(2+3j)x + (3+2j)y = 1$$
$$2x + 3jx + 3y + 2jy = 1$$
$$(2x+3y) + (3x+2y)j = 1 + 0j$$
$$\therefore$$
$$2x + 3y = 1$$
$$3x + 2y = 0.$$

This system of equation is solved, for example, by Cramer's rule as

$$\left.\begin{aligned}\Delta &= \begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix} = 4-9 = -5 \neq 0\\ \Delta x &= \begin{vmatrix}1 & 3\\ 0 & 2\end{vmatrix} = 2-0 = 2\\ \Delta y &= \begin{vmatrix}2 & 1\\ 3 & 0\end{vmatrix} = 0-3 = -3\end{aligned}\right\} \quad \therefore \quad x = \frac{\Delta x}{\Delta} = -\frac{2}{5} \quad \text{and} \quad y = \frac{\Delta y}{\Delta} = \frac{3}{5}.$$

6.75. Given

$$z^2 = 5 + 12j$$

Method 1: First, convert z^2 to polar form, and then calculate z, as

$$5 + 12j \quad \therefore \quad |5 + 12j| = \sqrt{5^2 + 12^2} = 13, \quad \arg(5 + 12j) = \arctan\frac{12}{5} \quad (I \text{ quadrant})$$

$$5 + 12j = 13\, e^{j \arctan(12/5)}$$

$$\therefore$$

$$z^2 = 13\, e^{j \arctan(12/5)} \quad \therefore \quad z = \pm\sqrt{13}\, e^{j \frac{\arctan(12/5)}{2}}$$

$$\therefore$$

$$z_1 = \sqrt{13}\left(\cos\left(\frac{\arctan(12/5)}{2}\right) + j \sin\left(\frac{\arctan(12/5)}{2}\right)\right) = 3 + 2j$$

$$z_2 = -\sqrt{13}\left(\cos\left(\frac{\arctan(12/5)}{2}\right) + j \sin\left(\frac{\arctan(12/5)}{2}\right)\right) = -3 - 2j$$

where the last calculation may be done numerically or with trigonometric identities that are not presented in this book.

Method 2: Being a complex number, algebraic form of z must be

$$z = a + bj, \quad (a, b) \in \mathbb{R}$$

therefore,

$$z^2 = a^2 + 2abj - b^2 = (a^2 - b^2) + (2ab)\, j.$$

By the equivalence of real and complex parts, it follows that

$$z^2 = 5 + 12j = (a^2 - b^2) + (2ab)\, j \quad \therefore \quad 2ab = 12 \quad \therefore \quad ab = 6 \Rightarrow \underline{a = \frac{6}{b}}$$

$$a^2 - b^2 = 5 \quad \therefore \quad \left(\frac{6}{b}\right)^2 - b^2 = 5.$$

The last equation is bi-quadratic, that is,

$$\frac{36}{b^2} - b^2 = 5 \;\; \therefore \;\; 36 - b^4 = 5b^2$$

$$b^4 + \underline{5b^2} \; - 36 = 0$$

$$b^4 + \underline{9b^2 - 4b^2} \; - 36 = 0$$

$$b^2(b^2+9) - 4(b^2+9) = 0$$

$$(b^2-4)(b^2+9) = 0$$

$$(b-2)(b+2)(b^2+9) = 0.$$

Note, due to $b \in \mathbb{R}$, that only real roots are acceptable; therefore,

$$b_{1,2} = \pm 2, \;\; \cancel{b_{3,4} \in \mathbb{C}} \;\; \therefore \;\; a = \frac{6}{b} = \frac{6}{\pm 2} = \pm 3$$

$$z_1 = 3 + 2j \;\; z_2 = -3 - 2j.$$

Therefore,

$$|z_{1,2}| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$\arg(z_1) = \text{atan2}\,(2,3)\,, \big(\text{in } I \text{ quadrant}, \;\; \approx 33.69^\circ\big)$$

$$\arg(z_2) = \text{atan2}\,(-2,-3)\,, \big(\text{in } III \text{ quadrant}, \;\; \approx \pi + 33.69^\circ \approx 213.69\big)$$

$$\therefore$$

$$z_1 = \sqrt{13}\,\big(\cos\big(\arctan(2/3)\big) + j \;\sin\big(\arctan(2/3)\big)\big)$$

$$z_2 = \sqrt{13}\,\big(-\cos\big(\arctan(2/3)\big) - j \;\sin\big(\arctan(2/3)\big)\big)\,.$$

6.76. One possible way to solve this equation may be as follows:

$$z^4 + 8 + 8\sqrt{3}\,j = 0 \;\; \therefore \;\; z^4 = -8 - 8\sqrt{3}\,j = |z_0|\,e^{j\,\varphi_0}$$

where $|z_0| = |-8 - 8\sqrt{3}\,j|$ and $\varphi_0 = \arg\big(-8 - 8\sqrt{3}\,j\big)$; thus,

$$-8 - 8\sqrt{3}\,j \;\; \therefore \;\; |z_0| = \sqrt{(-8)^2 + (-8\sqrt{3})^2} = \sqrt{64 + 3 \times 64} = \sqrt{4 \times 64} = \sqrt{4}\,\sqrt{64} = 16$$

$$\tan\varphi_0 = \frac{\text{Im}\,(z)}{\text{Re}\,(z)} = \frac{\cancel{-8} - \sqrt{3}}{\cancel{-8} - 1} = \sqrt{3} \quad \text{or}$$

$$\varphi = \text{atan2}\left(-\sqrt{3}, -1\right) \;\; \big(\text{in } III \text{ quadrant}\big)$$

$$\arg(z_0) = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \equiv -\frac{2\pi}{3}$$

$$\therefore \;\; z^4 = 16\,e^{j\,\frac{4\pi}{3}}.$$

Note that being $fourth$-order polynomial, it is necessary to correctly "unfold" the total of four distinct solutions for z. One possible method may be as follows $\left\{\sqrt{x^2} = \pm|x|\right\}$:

$$z^4 = 16\,e^{j\,\frac{4\pi}{3}} \quad \therefore \quad z_{1,2}^2 = \pm\left(16\,e^{j\,\frac{4\pi}{3}}\right)^{\frac{1}{2}} = \pm 4\,e^{j\,\frac{2\pi}{3}}$$

$$z_1^2 = 4\,e^{j\,\frac{2\pi}{3}}$$

$$z_2^2 = -4\,e^{j\,\frac{2\pi}{3}} = e^{j\pi}\,4\,e^{j\,\frac{2\pi}{3}} = 4\,e^{j\,(\pi+\frac{2\pi}{3})} = 4\,e^{j\,\frac{5\pi}{3}}$$

and, again,

$$z_1 = \pm\left(4\,e^{j\,\frac{2\pi}{3}}\right)^{\frac{1}{2}} = \pm 2\,e^{j\,\frac{\pi}{3}}$$

$$z_2 = \pm\left(4\,e^{j\,\frac{5\pi}{3}}\right)^{\frac{1}{2}} = \pm 2\,e^{j\,\frac{5\pi}{6}}.$$

So,

$$z_1 = 2\,e^{j\,\frac{\pi}{3}}$$

$$z_3 = -2\,e^{j\,\frac{\pi}{3}} = e^{j\,\pi}\,2\,e^{j\,\frac{\pi}{3}} = 2\,e^{j\,\frac{4\pi}{3}}$$

$$z_2 = 2\,e^{j\,\frac{5\pi}{6}}$$

$$z_4 = -2\,e^{j\,\frac{5\pi}{6}} = e^{j\,\pi}\,2\,e^{j\,\frac{5\pi}{6}} = 2\,e^{j\,\frac{11\pi}{6}}.$$

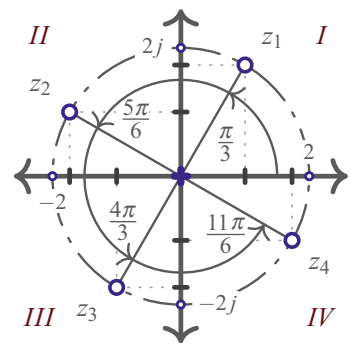
Closer inspection of the four roots shows that their arguments are symmetrically distributed on the circle perimeter, that is to say, step between any two roots must be $\pi/2$ (see Fig. 6.7). So, starting with z_1, general form of the roots may be expressed as

$$z_k = 2\,e^{j\,\left(\frac{\pi}{3}+k\,\frac{\pi}{2}\right)}$$

$$= 2\left[\cos\left(\frac{\pi}{3}+k\,\frac{\pi}{2}\right) + j\,\sin\left(\frac{\pi}{3}+k\,\frac{\pi}{2}\right)\right]$$

where, $k = 0, 1, 2, 3$.

Fig. 6.7 Example P.6.76



This kind of root distribution is general. For example, in the case of second-order equation, its two roots are separated by $2\pi/2 = \pi$ angle. Or if third-order equation, its three roots are separated by $2\pi/3$, etc.

In summary,

$$\underline{k=0:}\quad z_1 = 2\left[\cos\left(\frac{\pi}{3}\right) + j\,\sin\left(\frac{\pi}{3}\right)\right] = 1 + j\sqrt{3}.$$

$$\underline{k=1:}\quad z_2 = 2\left[\cos\left(\frac{\pi}{3}+\frac{\pi}{2}\right) + j\,\sin\left(\frac{\pi}{3}+\frac{\pi}{2}\right)\right] = -\sqrt{3} + j.$$

$$\underline{k=2:}\quad z_3 = 2\left[\cos\left(\frac{\pi}{3}+\pi\right) + j\,\sin\left(\frac{\pi}{3}+\pi\right)\right] = -1 - j\sqrt{3}.$$

$$\underline{k=3:}\quad z_4 = 2\left[\cos\left(\frac{\pi}{3}+\frac{3\pi}{2}\right) + j\,\sin\left(\frac{\pi}{3}+\frac{3\pi}{2}\right)\right] = \sqrt{3} - j.$$

Proof: Each of the four solutions $z_{1,2,3,4}$ should be verified, for example,

$$z_1^4 + 8 + 8\sqrt{3}\,i = 0$$

$$\left(2\,e^{i\left(\frac{5\pi}{6}\right)}\right)^4 + 8 + 8\sqrt{3}\,i = 0$$

$$16\,e^{i\left(\frac{10\pi}{3}\right)} + 8 + 8\sqrt{3}\,i = 0$$

$$16\,e^{i\left(\frac{6\pi}{3}\,0 + \frac{4\pi}{3}\right)} + 8 + 8\sqrt{3}\,i = 0$$

$$16\left[\cos\left(\frac{4\pi}{3}\right) + i\,\sin\left(\frac{4\pi}{3}\right)\right] + 8 + 8\sqrt{3}\,i = 0$$

$$16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right) + 8 + 8\sqrt{3}\,i = 0$$

$$-8 - 8\sqrt{3}\,i + 8 + 8\sqrt{3}\,i = 0$$

$$0 = 0\ \checkmark.$$

6.77. This kind of equation may be solved, for example, by enforcing the same form on both sides of the equation, as

$$\frac{1}{x+jy} = \frac{1}{2+j} + \frac{1}{-2+4j}$$

$$= \frac{-2+4j+2+j}{(2+j)(-2+4j)} = \frac{5j}{-4+8j-2j-4} = \frac{5j}{-8+6j} = \frac{1}{\frac{-8}{5j}+\frac{6j}{5j}} = \frac{1}{\frac{6}{5}-\frac{8}{5j}\frac{j}{j}}$$

$$\therefore$$

$$\frac{1}{x+j\,y} = \frac{1}{\frac{6}{5}+j\,\frac{8}{5}}.$$

Therefore, by inspection of left and right equation sides,

$$x = \frac{6}{5} \quad \text{and} \quad y = \frac{8}{5}.$$

Bode Plot

7

Decibel unit is used as a *relative* measure G between two quantities A_1 and A_2. There are two types of quantities: a quantity proportional either to power or to voltage. The former has multiplying factor "10", while the latter has multiplying factor "20", as

$$G_{\text{dB}} \stackrel{\text{def}}{=} 10 \log \frac{A_2}{A_1} \quad \text{or,} \quad G_{\text{dB}} \stackrel{\text{def}}{=} 20 \log \frac{A_2}{A_1}.$$

"dBm" unit is used as an *absolute* measure G of quantity A_2 normalized to 10^{-3} (i.e., "milli" on the engineering scale):

$$G_{\text{dBm}} \stackrel{\text{def}}{=} 10 \log \frac{A_2}{10^{-3}} \quad \text{or,} \quad G_{\text{dBm}} \stackrel{\text{def}}{=} 20 \log \frac{A_2}{10^{-3}}.$$

Decibel scale (power) notes the relationship between number of zeros in the ratio and the calculated value.

Ratio	Calculation	[dB]	Ratio	Calculation	[dB]
⋮	⋮	⋮	⋮	⋮	⋮
1/1000	$10\ \log(1/1000) = 10 \times (-3)$	-30	1/8	$10\ \log(1/8) = 10 \times (-0.9)$	-9
1/100	$10\ \log(1/100) = 10 \times (-2)$	-20	1/4	$10\ \log(1/4) = 10 \times (-0.6)$	-6
1/10	$10\ \log(1/10) = 10 \times (-1)$	-10	1/2	$10\ \log(1/2) = 10 \times (-0.3)$	-3
1	$10\ \log(1/1) = 10 \times (0)$	0	1	$10\ \log(1/1) = 10 \times (0)$	0
10	$10\ \log(10) = 10 \times (1)$	10	2	$10\ \log(2) = 10 \times (0.3)$	3
100	$10\ \log(100) = 10 \times (2)$	20	4	$10\ \log(4) = 10 \times (0.6)$	6
1000	$10\ \log(1000) = 10 \times (3)$	30	8	$10\ \log(8) = 10 \times (0.9)$	9
⋮	⋮	⋮	⋮	⋮	⋮

Note that each *ten* times increase translates to addition of 10 dB for quantities proportional to power or to 20 dB for quantities proportional to voltage. In addition, each division by factor 10 translates to addition of -10 dB for quantities proportional to power or to -20 dB for quantities proportional to voltage—thus, the common expression " -20 dB per decade". Similarly, each 2 times increase

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023

R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41200-4_7

translates to addition of 3 dB for quantities proportional to power or to 6 dB for quantities proportional to voltage. In addition, each division by factor 2 translates to addition of -3 dB for quantities proportional to power or to -6 dB for quantities proportional to voltage.

Basic building blocks of transfer functions: In engineering, instead of using letter i as the complex number, letter j is used instead (the "i" is already used to indicate AC current). In this book, there is no distinction between the two. General forms of four basic, first-order building blocks $H(jx)$ of complex transfer functions are

$$H_1(jx) = a_0 \tag{7.1}$$

$$H_2(jx) = j\frac{x}{x_0} \tag{7.2}$$

$$H_3(jx) = 1 \pm j\frac{x}{x_0} \tag{7.3}$$

$$H_4(jx) = \frac{1}{1 \pm j\frac{x}{x_0}} \tag{7.4}$$

where a_0, $x0 = \texttt{const.} \in \Re$ and $j^2 \equiv i^2 = -1$. As a rule of a thumb, the phase Bode plot covers a four-decade interval centered around x_0. For example, if $x_0 = 1 = 10^0$, then two decades above are 10^1 and 10^2, and two decades below are 10^{-1} and 10^{-2}, which makes the interval from 0.01 to 100.

Problems

7.1 Basic Complex Functions

Study and sketch piecewise linear approximation Bode plots of complex functions in P.7.1 to P.7.12. Assume that quantity x is proportional to voltage.

7.1. $z(x) = -2$

7.2. $z(x) = \frac{1}{10}$

7.3. $z(x) = j\frac{x}{2}$

7.4. $z(x) = j\frac{x}{10}$

7.5. $z(x) = 1 + j\frac{x}{2}$

7.6. $z(x) = 1 + j\frac{x}{10}$

7.7. $z(x) = 1 - j\frac{x}{2}$

7.8. $z(x) = 1 - j\frac{x}{10}$

7.9. $z(x) = \frac{1}{1 + j\frac{x}{2}}$

7.10. $z(x) = \frac{1}{1 + j\frac{x}{10}}$

7.11. $z(x) = \frac{1}{1 - j\frac{x}{2}}$

7.12. $z(x) = \frac{1}{1 - j\frac{x}{10}}$

7.2 Bode Plot Examples

Sketch Bode plot of functions in P.7.13 to P.7.16. Assume that quantity x is proportional to voltage.

7.13. $z(x) = (1 - jx)\left(1 + j\dfrac{x}{100}\right)$

7.14. $z(x) = 500\,\dfrac{1 + jx}{100 - jx}$

7.15. $z(x) = 100\,\dfrac{10 + jx}{100 + jx}$

7.16. $z(x) = 20\,000\,\dfrac{2 + jx}{220\,jx + 4\,000 - x^2}$

Answers

7.1 Basic Complex Functions

7.1. Given $z(x) = -2$, Bode plot of this function may be derived as follows:

(a) *Module (gain) limits*: The general form is (7.1), and a real constant is not a function of any variable; thus,

$$|z(x)| = |-2| = 2$$

$$\therefore$$

$$\lim_{x\to 0} |z(x)| = \lim_{x\to 0} 2 = 2 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 2 = 6\,\text{dB}$$

$$\lim_{x\to 2} |z(x)| = \lim_{x\to 2} 2 = 2 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 2 = 6\,\text{dB}$$

$$\lim_{x\to +\infty} |z(x)| = \lim_{x\to +\infty} 2 = 2 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 2 = 6\,\text{dB}$$

which is to say that module graph is a constant 6 dB line (see Fig. 7.1 (left, top)).

(b) *Phase limits*: The imaginary part $\Im(z)$ equal to zero, and the real part $\Re(z)$ is a negative number; therefore, by definition (see Chap. 6),

$$\theta(z) = \arctan\left[\frac{\Im(z)}{\Re(z)}\right] = \text{atan2}\,(0, -2) = 180^\circ = \pm\pi = \text{const.}$$

Therefore, its phase graph $\theta(z)$ is a constant line equal to π (see Fig. 7.1 (left, bottom)).

(c) *Bode plot interpretation*: A real constant (a.k.a. "gain") is not a function of variable x. Negative gain, however, indicates the phase inversion, thus $\theta(z) = \pi$, as in the well-known inverting amplifier.

7.2. Given $z(x) = 1/10$, Bode plot of this function may be derived as follows:

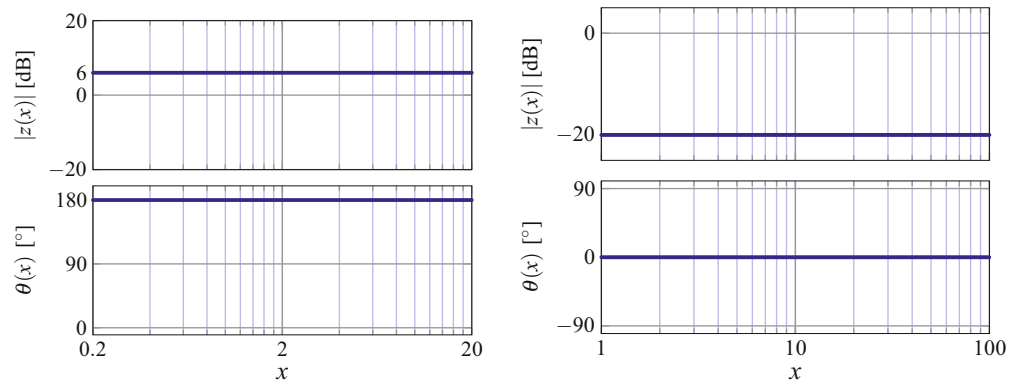


Fig. 7.1 Example P.7.1

(a) *Module (gain) limits*: The general form is (7.1), and a real constant is not a function of any variable; thus,

$$|z(x)| = \left|\frac{1}{10}\right| = \frac{1}{10} = 10^{-1}$$

$$\therefore$$

$$\lim_{x\to 0}|z(x)| = \lim_{x\to 0}\frac{1}{10} = \frac{1}{10} \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 10^{-1} = -20\log 10 = -20\,\text{dB}$$

$$\lim_{x\to 2}|z(x)| = \lim_{x\to 2}\frac{1}{10} = \frac{1}{10} \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 10^{-1} = -20\log 10 = -20\,\text{dB}$$

$$\lim_{x\to +\infty}|z(x)| = \lim_{x\to +\infty}\frac{1}{10} = \frac{1}{10} \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 10^{-1} = -20\log 10 = -20\,\text{dB}$$

which is to say that module graph is a constant -20 dB line (see Fig. 7.1 (right, top)).

(b) *Phase limits*: Imaginary part $\Im(z)$ equals zero, and real part $\Re(z)$ is a positive number; therefore, by definition (see Chap. 6),

$$\theta(z) = \arctan\left[\frac{\Im(z)}{\Re(z)}\right] = \text{atan2}\,(0,\, 1/10) = 0^\circ = \text{const.}.$$

Therefore, its phase graph $\theta(z)$ is a constant line equal to 0° (see Fig. 7.1 (right, bottom)).

(c) *Bode plot interpretation*: A real constant (a.k.a. "gain") is not a function of variable x; fractional gain, however, is equivalent to negative gain if measured in dB units. Positive gain means that there is no phase inversion, thus $\theta(z) = 0^\circ$, as in the well-known non-inverting amplifier.

7.3. Given $z(x) = j^x/2$, Bode plot of this function may be derived as follows:

(a) 0 dB *point*: By comparison with the general form (7.2), $x_0 = 2$.

(b) *Module limits*: Real part $\Re(z)$ equals zero; thus,

$$|z(x)| = \left|j\frac{x}{2}\right| = \frac{x}{2}$$

$$\therefore$$

$$\lim_{x\to 0}|z(x)| = \lim_{x\to 0}\frac{x}{2} = 0 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 0 = -\infty$$

$$\lim_{x\to 2}|z(x)| = \lim_{x\to 2}\frac{x}{2} = 1 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 1 = 0\,\text{dB}$$

$$\lim_{x\to +\infty}|z(x)| = \lim_{x\to +\infty}\frac{x}{2} = +\infty \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log(+\infty) = +\infty.$$

(c) *Module (gain) plot*: In log-log scale, this function (in decibels) is linear. Thus, in order to determine linear function, it is sufficient to calculate coordinates of two points. Convenient choice is to calculate $z(x)$ at $x = 0.2$ (ten times smaller) and $x = 20$ (ten times greater) than $x_0 = 2$ as

$$z(x = 0.2) = 20\log\left(\frac{0.2}{2}\right) = 20\log\left(\frac{1}{10}\right) = -20\,\text{dB}$$

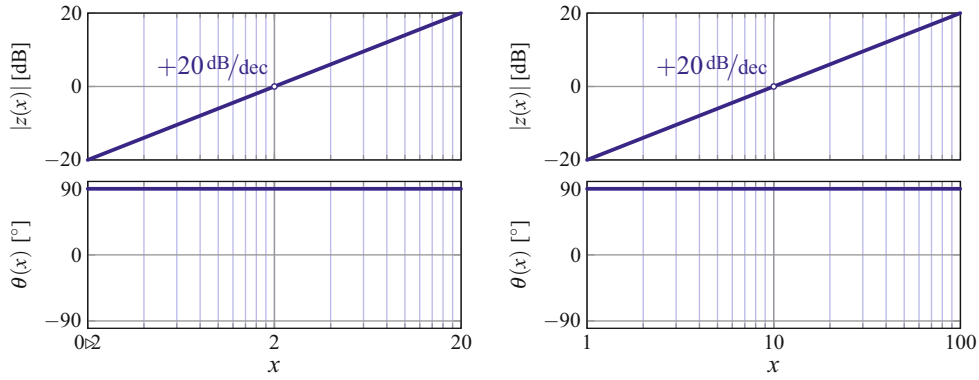


Fig. 7.2 Example P.7.3

$$z(x = 20) = 20 \log \left(\frac{20}{2} \right) = 20 \log (10) = +20\,\text{dB}$$

which is to say that gain increases 20 dB for each tenfold increase of x (Fig. 7.2 (left, top)).

(d) *Phase limits*: Real part $\Re(z)$ equals zero, and the imaginary part $\Im(z)$ is positive number; thus, in the first quadrant by definition (see Chap. 6)

$$\theta(z) = \arctan \left[\frac{\Im(z)}{\Re(z)} \right] = \text{atan2}\,(x/2,\, 0) = \arctan \left(\frac{x/2}{0} \right) = \frac{\pi}{2} = \text{const.}.$$

Therefore, its phase graph $\theta(z)$ is a constant line equal to $\pi/2$ (see Fig. 7.2 (left, bottom)).

(e) *Bode plot interpretation*: Gain of this function, as measured in dB, is directly proportional to the variable x. For any tenfold increase of x, there is "20 dB/decade" gain. Phase, however, is constant and fixed to $\pi/2$ (i.e., the phase of $+j$).

7.4. Given $z(x) = jx/10$, Bode plot of this function may be derived as follows:

(a) 0 dB *point*: By comparison with the general form (7.2), $x_0 = 10$.

(b) *Module (gain) limits*: Real part $\Re(z)$ equals zero; thus,

$$|z(x)| = \left| j \frac{x}{10} \right| = \frac{x}{10}$$

$$\therefore$$

$$\lim_{x \to 0} |z(x)| = \lim_{x \to 0} \frac{x}{10} = 0 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20 \log 0 = -\infty$$

$$\lim_{x \to 10} |z(x)| = \lim_{x \to 10} \frac{x}{10} = 1 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20 \log 1 = 0\,\text{dB}$$

$$\lim_{x \to +\infty} |z(x)| = \lim_{x \to +\infty} \frac{x}{10} = +\infty \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20 \log(+\infty) = +\infty.$$

(c) *Module (gain) plot*: In log-log scale, this function (in decibels) is linear. Thus, in order to determine linear function, it is sufficient to calculate coordinates of two points. Convenient choice is to calculate $z(x)$ at $x = 1$ (ten times smaller) and $x = 100$ (ten times greater) than $x_0 = 10$ as,

$$z(x = 1) = 20 \log \left(\frac{1}{10} \right) = -20\,\text{dB}$$

$$z(x = 100) = 20 \log \left(\frac{100}{10} \right) = 20 \log (10) = +20\,\text{dB}$$

which is to say that module increases 20 dB for each tenfold increase of x (Fig. 7.2 (right, top)).

(d) *Phase limits*: Real part $\Re(z)$ equals zero, and imaginary part $\Im(z)$ is a positive number; by definition (see Chap. 6),

$$\theta(z) = \arctan \left[\frac{\Im(z)}{\Re(z)} \right] = \text{atan2} \left({}^{x}/_{10}, 0 \right) = \arctan \left(\frac{{}^{x}/_{10}}{0} \right) = \frac{\pi}{2} = \text{const.}$$

Therefore, its phase graph $\theta(z)$ is a constant line equal to $\pi/2$ (see Fig. 7.2 (right, bottom)).

7.5. Given $z(x) = 1 + j{}^{x}/_{2}$ where both real and imaginary parts are positive, module and phase of a complex function (see Chap. 6) that make Bode plot may be derived as follows:

(a) 3 dB *point*: By comparison with the general form (7.3), $x_0 = 2$.

(b) *Module limits*: Both real and imaginary parts are present; thus,

$$|z(x)| = \left| 1 + j \frac{x}{2} \right| = 20 \log \sqrt{1 + \left(\frac{x}{2} \right)^2} \tag{7.5}$$

$$\therefore$$

$$\lim_{x \to 0} |z(x)| = \lim_{x \to 0} \sqrt{1 + \left(\frac{x}{2} \right)^2} = 1 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20 \log 1 = 0$$

$$\lim_{x \to 2} |z(x)| = \lim_{x \to 2} \sqrt{1 + \left(\frac{x}{2} \right)^2} = \sqrt{2} \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20 \log \sqrt{2} = +3\,\text{dB}$$

$$\lim_{x \to +\infty} |z(x)| = \lim_{x \to +\infty} \sqrt{1 + \left(\frac{x}{2} \right)^2} = +\infty \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20 \log(+\infty) = +\infty.$$

(c) *Module (gain) plot*—Hand analysis of nonlinear function (7.5) is done by piecewise linear approximation as follows:

(a) $\underline{x \to 0}$: $|z(x)|_{\text{dB}} = 0\,\text{dB}$.

(b) $\underline{x = 0.2}$: i.e., at 1/10 of x_0 or less

$$|z(x)|_{\text{dB}} = 20 \log \sqrt{1 + \left(\frac{0.2}{2} \right)^2} = 20 \log \sqrt{1 + \left(\frac{1}{10} \right)^2}$$

$$= \left\{ 1 \gg \frac{1}{10} \quad \Rightarrow \quad 1 + \left(\frac{1}{10} \right)^2 \approx 1 \right\}$$

$$\approx 20 \log 1 = 0\,\text{dB}.$$

(c) $\underline{x \gg x_0}$: That is to say, at ten times of x_0 or more, $x/x_0 \gg 1 \;\Rightarrow\; 1 + (x/x_0)^2 \approx (x/x_0)^2$

$$|z(x)|_{\text{dB}} = 20\log\sqrt{1+\left(\frac{x}{x_0}\right)^2} \approx 20\log\sqrt{\left(\frac{x}{x_0}\right)^2} \approx 20\log\left|\frac{x}{x_0}\right|$$
$$= \left\{\log\frac{a}{b} = \log a - \log b\right\} = 20\log|x| - \underbrace{20\log x_0}_{\text{const.}}$$

In conclusion, for $x \gg x_0$, module $|z(x)|_{\text{dB}}$ function asymptotically follows function x (on log scale) that crosses horizontal axis at $x = x_0$.

Therefore, piecewise linear approximation of $z(x) = 1 + jx/2$ is

$$\underline{0 < x \leq 2} \;: |z(x)|_{\text{dB}} \approx 0$$
$$\underline{x = 2} \;: |z(x)|_{\text{dB}} = +3\text{dB} \;\; \text{(this is the exact value)}$$
$$\underline{x > 2} \;: |z(x)|_{\text{dB}} \approx x \;\; \text{(slope)}$$

which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 3 dB at $x = 2$ point (Fig. 7.3 (left, top)).

(d) *Phase limits*: Both real and imaginary parts are present; thus, in the first quadrant,

$$\theta(z) = \arctan\left[\frac{\Im(z)}{\Re(z)}\right] = \text{atan2}\,(x/2,\, 1) = \arctan\left(\frac{x}{2}\right) \tag{7.6}$$

$\therefore$

$$\lim_{x\to 0}\theta(x) = \lim_{x\to 0}\arctan\left(\frac{x}{2}\right) = \arctan(0) = 0^\circ$$
$$\lim_{x\to 2}\theta(x) = \lim_{x\to 2}\arctan\left(\frac{x}{2}\right) = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4} = 45^\circ$$
$$\lim_{x\to +\infty}\theta(x) = \lim_{x\to +\infty}\arctan\left(\frac{x}{2}\right) = \arctan(\infty) = \frac{\pi}{2} = 90^\circ.$$

(e) *Phase plot*—Hand analysis of nonlinear function (7.6) is done by piecewise linear approximation as follows:

(a) $\underline{x \to 0}$: $\theta(x) = 0^\circ$.

(b) $\underline{x = 0.2}$: i.e., at 1/10 of x_0

$$\theta(x = 0.2) = \text{atan2}\,(0.2,\, 2) = \arctan(0.1) = 5.7^\circ \approx 0^\circ.$$

(c) $\underline{x = 20}$: i.e., at 10 times of x_0

$$\theta(x = 20) = \text{atan2}\,(20,\, 2) = \arctan(10) = 84.3^\circ \approx -90^\circ = \frac{\pi}{2}.$$

Therefore, piecewise linear approximation of $\theta(x) = \arctan(x/2)$ is

$$\underline{0 < x \leq 0.2} \;: \theta(x) \approx 0$$

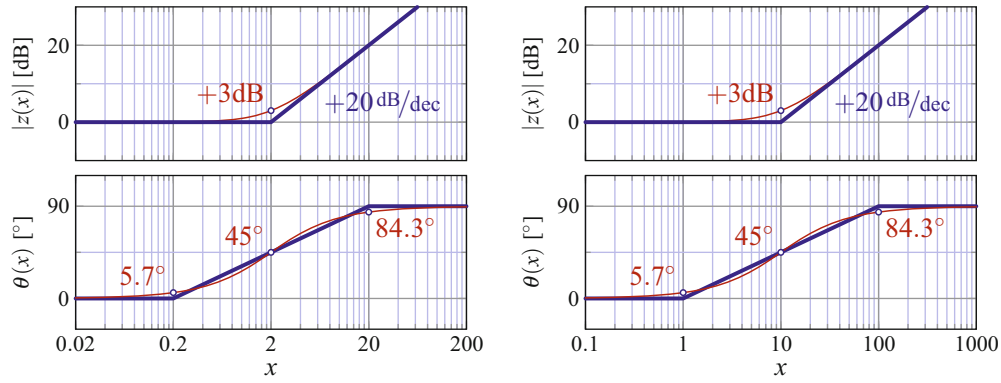


Fig. 7.3 Example P.7.5

$$\underline{x = 0.2} \ : \theta(x) = 5.7^\circ \approx 0$$

$$\underline{x = 2} \ : \theta(x) = 45^\circ = \frac{\pi}{4} \quad \text{(this is the exact value)}$$

$$\underline{x = 20} \ : \theta(x) = 84.3^\circ \approx 90^\circ$$

$$\underline{x > 20} \ : \theta(x) \approx 90^\circ$$

which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 5.7° at $x = x_0/10$ and $10\,x_0$ points (Fig. 7.3 (left, bottom)). Note that $\theta = 45^\circ$ is found at $x = x_0$.

7.6. Given $z(x) = 1 + jx/10$, module and phase of a complex function (see Chap. 6) that make Bode plot may be derived as follows:

(a) 3 dB *point*: By comparison with the general form (7.3), $x_0 = 10$.
(b) *Module limits*: Both real and imaginary parts are present; thus,

$$|z(x)| = \left|1 + j\frac{x}{10}\right| = 20\log\sqrt{1 + \left(\frac{x}{10}\right)^2} \tag{7.7}$$

$$\therefore$$

$$\lim_{x\to 0}|z(x)| = \lim_{x\to 0}\sqrt{1 + \left(\frac{x}{10}\right)^2} = 1 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 1 = 0\,\text{dB}$$

$$\lim_{x\to 10}|z(x)| = \lim_{x\to 10}\sqrt{1 + \left(\frac{x}{10}\right)^2} = \sqrt{2} \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log\sqrt{2} = +3\,\text{dB}$$

$$\lim_{x\to +\infty}|z(x)| = \lim_{x\to +\infty}\sqrt{1 + \left(\frac{x}{10}\right)^2} = +\infty \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log(+\infty) = +\infty.$$

(c) *Module plot*—Hand analysis of nonlinear function (7.7) is done by piecewise linear approximation as follows:

(a) $\underline{x \to 0}$: $|z(x)|_{\text{dB}} = 0\,\text{dB}$.

(b) $\underline{x = 1}$: i.e., at $1/10$ of x_0 or less

$$|z(x)|_{\text{dB}} = 20\log\sqrt{1+\left(\frac{1}{10}\right)^2} = \left\{1 \gg \frac{1}{10} \;\Rightarrow\; 1+\left(\frac{1}{10}\right)^2 \approx 1\right\}$$
$$\approx 20\log 1 = 0\,\text{dB}.$$

(c) $\underline{x \gg x_0}$: At ten times of x_0 or more, that is to say, $x/x_0 \gg 1 \;\Rightarrow\; 1 + (x/x_0)^2 \approx (x/x_0)^2$

$$|z(x)|_{\text{dB}} = 20\log\sqrt{1+\left(\frac{x}{x_0}\right)^2} \approx 20\log\sqrt{\left(\frac{x}{x_0}\right)^2} \approx 20\log\left|\frac{x}{x_0}\right|$$
$$= \left\{\log\frac{a}{b} = \log a - \log b\right\} = 20\log|x| - \underbrace{20\log x_0}_{\text{const.}}$$

In conclusion, for $x \gg x_0$, module $|z(x)|_{\text{dB}}$ function asymptotically follows function x (on log scale) that crosses horizontal axis at $x = x_0$.

Therefore, piecewise linear approximation of $z(x) = 1 + jx/10$ is

$$\underline{0 \geq x \leq 10} \;: |z(x)|_{\text{dB}} \approx 0$$
$$\underline{x = 10} \;: |z(x)|_{\text{dB}} = +3\text{dB} \;\; \text{(this is the exact value)}$$
$$\underline{x > 10} \;: |z(x)|_{\text{dB}} \approx x \;\; \text{(slope)}$$

which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 3 dB at $x = 10$ point (Fig. 7.3 (right, top)).

(d) *Phase limits*: Both real and imaginary parts are positive; thus,

$$\theta(z) = \arctan\left[\frac{\Im(z)}{\Re(z)}\right] = \text{atan2}\,(x/10,\, 1) = \arctan\left(\frac{x}{10}\right) \tag{7.8}$$

$\therefore$

$$\lim_{x\to 0}\theta(x) = \lim_{x\to 0}\arctan\left(\frac{x}{10}\right) = \arctan(0) = 0^\circ$$
$$\lim_{x\to 10}\theta(x) = \lim_{x\to 10}\arctan\left(\frac{x}{10}\right) = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4} = 45^\circ$$
$$\lim_{x\to +\infty}\theta(x) = \lim_{x\to +\infty}\arctan\left(\frac{x}{10}\right) = \arctan(\infty) = \frac{\pi}{10} = 90^\circ.$$

(e) *Phase plot*—Hand analysis of nonlinear function (7.8) is done by piecewise linear approximation as follows:

(a) $\underline{x \to 0}$: $\theta(x) = 0^\circ$.

(b) $\underline{x = 1}$: i.e., at $1/10$ of x_0

$$\theta(x=1) = \arctan\left(\frac{1}{10}\right) = 5.7^\circ \approx 0^\circ.$$

(c) $\underline{x = 100}$: i.e., at 10 times of x_0

$$\theta(x=100) = \arctan\left(\frac{100}{10}\right) = \arctan(10) = 84.3^\circ \approx 90^\circ = \frac{\pi}{2}.$$

Therefore, piecewise linear approximation of $\theta(x) = \arctan(x/10)$ is

$$\begin{aligned}
\underline{0 < x \leq 1} &: \theta(x) \approx 0 \\
\underline{x = 1} &: \theta(x) = 5.7^\circ \approx 0 \\
\underline{x = 10} &: \theta(x) = 45^\circ = \frac{\pi}{4} \quad \text{(this is the exact value)} \\
\underline{x = 100} &: \theta(x) = 84.3^\circ \approx 90^\circ \\
\underline{x > 100} &: \theta(x) \approx 90^\circ
\end{aligned}$$

which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 5.7° at $x = x_0/10$ and $10\,x_0$ points (Fig. 7.3 (right, bottom)). Note that $\theta = 45^\circ$ is found at $x = x_0$.

7.7. Given $z(x) = 1 - jx/2$, module and phase of a complex function (see Chap. 6) that make Bode plot may be derived as follows:

(a) 3 dB *point*: By comparison with the general form (7.3), $x_0 = 2$.
(b) *Module limits*: Real part is positive and imaginary part is negative; thus,

$$|z(x)| = \left|1 - j\frac{x}{2}\right| = 20\log\sqrt{1 + \left(\frac{x}{2}\right)^2} \tag{7.9}$$

$$\therefore$$

$$\lim_{x\to 0}|z(x)| = \lim_{x\to 0}\sqrt{1+\left(\frac{x}{2}\right)^2} = 1 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 1 = 0\text{dB}$$

$$\lim_{x\to 2}|z(x)| = \lim_{x\to 2}\sqrt{1+\left(\frac{x}{2}\right)^2} = \sqrt{2} \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log\sqrt{2} = +3\,\text{dB}$$

$$\lim_{x\to +\infty}|z(x)| = \lim_{x\to +\infty}\sqrt{1+\left(\frac{x}{2}\right)^2} = +\infty \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log(+\infty) = +\infty.$$

(c) *Module (gain) plot*—Hand analysis of nonlinear function (7.9) is done by piecewise linear approximation as follows:
(a) $\underline{x \to 0}$: $|z(x)|_{\text{dB}} = 0\,\text{dB}$.
(b) $\underline{x = 0.2}$: i.e., at 1/10 of x_0

$$|z(x)|_{\text{dB}} = 20\log\sqrt{1+\left(\frac{0.2}{2}\right)^2} = 20\log\sqrt{1+\left(\frac{1}{10}\right)^2}$$

$$= \left\{ 1 \gg \frac{1}{10} \;\Rightarrow\; 1 + \left(\frac{1}{10}\right)^2 \approx 1 \right\}$$

$$\approx 20 \log 1 = 0\,\text{dB}.$$

(c) $\underline{x \gg 2}$: At ten times of x_0 or more, that is to say, $x/x_0 \gg 1 \;\Rightarrow\; 1 + (x/x_0)^2 \approx (x/x_0)^2$

$$|z(x)|_{\text{dB}} = 20 \log \sqrt{1 + \left(\frac{x}{x_0}\right)^2} \approx 20 \log \sqrt{\left(\frac{x}{x_0}\right)^2} \approx 20 \log \left|\frac{x}{x_0}\right|$$

$$= \left\{ \log \frac{a}{b} = \log a - \log b \right\} = 20 \log |x| - \underbrace{20 \log x_0}_{\text{const.}}$$

In conclusion, for $x \gg x_0$, module $|z(x)|_{\text{dB}}$ function asymptotically follows function x (on log scale) that crosses horizontal axis at $x = x_0$:

Therefore, piecewise linear approximation of $z(x) = 1 - jx/2$ is

$$\underline{0 < x \leq 2} \;: |z(x)|_{\text{dB}} \approx 0\,\text{dB}$$

$$\underline{x = 2} \;: |z(x)|_{\text{dB}} = +3\text{dB} \quad \text{(this is the exact value)}$$

$$\underline{x >\!\!> 2} \;: |z(x)|_{\text{dB}} \approx x \quad \text{(slope)}$$

which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 3 dB at $x = 2$ point (Fig. 7.4 (left, top)).

(d) *Phase limits*: Real part is positive and imaginary part is negative; thus, in third quadrant

$$\theta(z) = \arctan\left[\frac{\Im(z)}{\Re(z)}\right] = \text{atan2}\,(-x/2,\, 1) = -\arctan\left(\frac{x}{2}\right) \tag{7.10}$$

$$\therefore$$

$$\lim_{x \to 0} \theta(x) = \lim_{x \to 0} \left(-\arctan\left(\frac{x}{2}\right)\right) = -\arctan(0) = 0^\circ$$

$$\lim_{x \to 2} \theta(x) = \lim_{x \to 2} \left(-\arctan\left(\frac{x}{2}\right)\right) = -\arctan\left(\frac{1}{1}\right) = -\frac{\pi}{4} = -45^\circ$$

$$\lim_{x \to +\infty} \theta(x) = \lim_{x \to +\infty} \left(-\arctan\left(\frac{x}{2}\right)\right) = -\arctan(\infty) = -\frac{\pi}{2} = -90^\circ.$$

(e) *Phase plot*—Hand analysis of nonlinear function (7.10) is done by piecewise linear approximation as follows:

(a) $\underline{x \to 0}$: $\theta(x) = 0^\circ$.

(b) $\underline{x = 0.2}$: i.e., at 1/10 of x_0

$$\theta(x = 0.2) = \left(-\arctan\left(\frac{0.2}{2}\right)\right) = -\arctan(0.1) = -5.7^\circ \approx 0.$$

(c) $\underline{x = 20}$: i.e., at 10 times of x_0

$$\theta(x=20) = -\arctan\left(\frac{20}{2}\right) = -\arctan(10) = -84.3^\circ \approx -90^\circ = -\frac{\pi}{2}.$$

Therefore, piecewise linear approximation of $\theta(x) = -\arctan(x/2)$ is

$$\begin{aligned}
\underline{0 < x \le 0.2} &: \theta(x) \approx 0^\circ \\
\underline{x = 0.2} &: \theta(x) = -5.7^\circ \approx 0^\circ \\
\underline{x = 2} &: \theta(x) = -45^\circ = -\frac{\pi}{4} \quad \text{(this is the exact value)} \\
\underline{x = 20} &: \theta(x) = -84.3^\circ \approx -90^\circ \\
\underline{x > 20} &: \theta(x) \approx -90^\circ
\end{aligned}$$

which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 5.7° at $x = x_0/10$ and $10\,x_0$ points (Fig. 7.4 (left, bottom)). Note that $\theta = -45^\circ$ is found at $x = x_0$.

7.8. Given $z(x) = 1 - j^x/_{10}$, module and phase of a complex function (see Chap. 6) that make Bode plot may be derived as follows:

(a) 3 dB *point*: By comparison with the general form (7.3), $x_0 = 10$.
(b) *Module limits*: Both real and imaginary parts are present; thus,

$$|z(x)| = \left|1 - j\frac{x}{10}\right| = 20\log\sqrt{1 + \left(\frac{x}{10}\right)^2} \tag{7.11}$$

$$\therefore$$

$$\lim_{x\to 0}|z(x)| = \lim_{x\to 0}\sqrt{1+\left(\frac{x}{10}\right)^2} = 1 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 1 = 0$$

$$\lim_{x\to 10}|z(x)| = \lim_{x\to 10}\sqrt{1+\left(\frac{x}{10}\right)^2} = \sqrt{2} \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log\sqrt{2} = +3$$

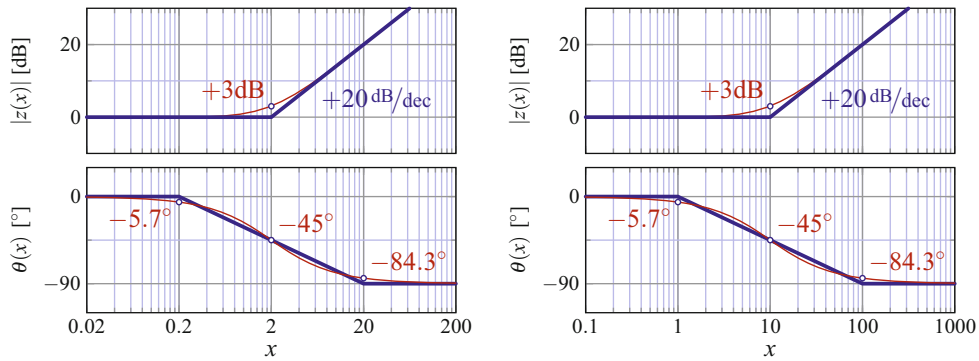


Fig. 7.4 Example P.7.7

$$\lim_{x\to+\infty}|z(x)| = \lim_{x\to+\infty}\sqrt{1+\left(\frac{x}{10}\right)^2} = +\infty \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log(+\infty) = +\infty.$$

(c) *Module plot*—Hand analysis of nonlinear function (7.11) is done by piecewise linear approximation as follows:

(a) $\underline{x \to 0}$: $|z(x)|_{\text{dB}} = 0$.

(b) $\underline{x = 1}$: i.e., at $1/10$ of x_0

$$|z(x)|_{\text{dB}} = 20\log\sqrt{1+\left(\frac{1}{10}\right)^2} = \left\{1 \gg \frac{1}{10} \quad \Rightarrow \quad 1+\left(\frac{1}{10}\right)^2 \approx 1\right\}$$
$$\approx 20\log 1 = 0.$$

(c) $\underline{x \gg 10}$: At ten times of x_0 or more, that is to say, $x/x_0 \gg 1 \quad \Rightarrow \quad 1 + (x/x_0)^2 \approx (x/x_0)^2$

$$|z(x)|_{\text{dB}} = 20\log\sqrt{1+\left(\frac{x}{x_0}\right)^2} \approx 20\log\sqrt{\left(\frac{x}{x_0}\right)^2} \approx 20\log\left|\frac{x}{x_0}\right|$$
$$= \left\{\log\frac{a}{b} = \log a - \log b\right\} = 20\log|x| - \underbrace{20\log x_0}_{\text{const.}}$$

In conclusion, for $x \gg x_0$, module $|z(x)|_{\text{dB}}$ function asymptotically follows function x (on log scale) that crosses horizontal axis at $x = x_0$.

Therefore, piecewise linear approximation of $z(x) = 1 - jx/10$ is

$$\underline{0 < x \le 1} \ : |z(x)|_{\text{dB}} \approx 0\,\text{dB}$$
$$\underline{1 \ge x \le 10} \ : |z(x)|_{\text{dB}} \approx 0\,\text{dB}$$
$$\underline{x = 10} \ : |z(x)|_{\text{dB}} = +3\text{dB} \quad \text{(this is the exact value)}$$
$$\underline{x > 10} \ : |z(x)|_{\text{dB}} \approx x$$

which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 3 dB at $x = 2$ point (Fig. 7.4 (right, top)).

(d) *Phase limits*: Real part is positive and imaginary part is negative; thus, in the fourth quadrant,

$$\theta(z) = \arctan\left[\frac{\Im(z)}{\Re(z)}\right] = \text{atan2}\,(-x/10,\, 1) = -\arctan\left(\frac{x}{10}\right) \tag{7.12}$$

$\therefore$

$$\lim_{x\to 0}\theta(x) = \lim_{x\to 0}\left(-\arctan\left(\frac{x}{10}\right)\right) = -\arctan(0) = 0^\circ$$
$$\lim_{x\to 10}\theta(x) = \lim_{x\to 10}\left(-\arctan\left(\frac{x}{10}\right)\right) = -\arctan(1) = -\frac{\pi}{4} = -45^\circ$$
$$\lim_{x\to+\infty}\theta(x) = \lim_{x\to+\infty}\left(-\arctan\left(\frac{x}{10}\right)\right) = -\arctan(\infty) = -\frac{\pi}{10} = -90^\circ.$$

(e) *Phase plot*—Hand analysis of nonlinear function (7.12) is done by piecewise linear approximation as follows:
(a) $\underline{x \to 0}$: $\theta(x) = 0°$.
(b) $\underline{x = 1}$: i.e., at $1/10$ of x_0

$$\theta(x = 1) = \arctan\left(-\frac{1}{10}\right) = -5.7° \approx 0°.$$

(c) $\underline{x = 100}$: i.e., at 10 times of x_0

$$\theta(x = 100) = -\arctan\left(\frac{100}{10}\right) = -\arctan(10) = -84.3° \approx -90° = -\frac{\pi}{2}.$$

Therefore, piecewise linear approximation of $\theta(x) = -\arctan(x/10)$ is

$$\begin{aligned}
\underline{0 < x \leq 1} &: \theta(x) \approx 0° \\
\underline{x = 1} &: \theta(x) = -5.7° \approx 0° \\
\underline{x = 10} &: \theta(x) = -45° = -\frac{\pi}{4} \quad \text{(this is the exact value)} \\
\underline{x = 100} &: \theta(x) = -84.3° \approx -90° \\
\underline{x > 100} &: \theta(x) \approx -90°
\end{aligned}$$

which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 5.7° at $x = x_0/10$ and $10\,x_0$ points (Fig. 7.4 (right, bottom)). Note that $\theta = -45°$ is found at $x = x_0$.

7.9. Given $z(x) = 1/(1 + j^x/_2)$, module and phase of a complex function that make Bode plot may be derived as follows:

(a) -3 dB *point*: By comparison with the general form (7.4), $x_0 = 2$.
(b) *Module limits*: This function is inverse complex; thus,

$$|z(x)| = \left|\frac{1}{1 + j\frac{x}{2}}\right| = \frac{1}{\sqrt{1 + \left(\frac{x}{2}\right)^2}} \tag{7.13}$$

$\therefore$

$$\lim_{x\to 0} |z(x)| = \lim_{x\to 0} \frac{1}{\sqrt{1 + \left(\frac{x}{2}\right)^2}} = 1 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20 \log 1 = 0$$

$$\lim_{x\to 2} |z(x)| = \lim_{x\to 2} \frac{1}{\sqrt{1 + \left(\frac{x}{2}\right)^2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20 \log \frac{1}{\sqrt{2}} = -3$$

$$\lim_{x\to+\infty}|z(x)| = \lim_{x\to+\infty}\frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = 0 \quad\Rightarrow\quad |z(x)|_{\text{dB}} = 20\log 0 = -\infty.$$

(c) *Module plot*—Hand analysis of nonlinear function (7.13) is done by piecewise linear approximation as follows:

(a) $\underline{x \to 0}$: $|z(x)|_{\text{dB}} = 0$.
(b) $\underline{x \ll x_0}$: That is to say, $x/x_0 \to 0 \quad\Rightarrow\quad 1 + (x/x_0)^2 \approx 1$

$$|z(x)|_{\text{dB}} = 20\log\frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}} \approx 20\log\frac{1}{\sqrt{1+0}} = \left\{\frac{1}{x} = x^{-1},\quad \log a^b = b\log a\right\}$$

$$\approx -20\log 1 = 0.$$

(c) $\underline{x \gg x_0}$: That is to say, $x/x_0 \gg 1 \quad\Rightarrow\quad 1 + (x/x_0)^2 \approx (x/x_0)^2$

$$|z(x)|_{\text{dB}} = 20\log\frac{1}{\sqrt{1+\left(\frac{x}{x_0}\right)^2}} \approx 20\log\frac{1}{\sqrt{\left(\frac{x}{x_0}\right)^2}} \approx 20\log\left|\frac{x_0}{x}\right|$$

$$= \left\{\log\frac{a}{b} = \log a - \log b\right\} = \underbrace{20\log x_0}_{\text{const.}} - 20\log|x|.$$

In conclusion, for $x \gg x_0$, module $|z(x)|_{\text{dB}}$ function asymptotically follows function $-x$ (on log scale) that crosses horizontal axis at $x = x_0$.

Therefore, piecewise linear approximation of $|z(x)| = |1/(1 + jx/2)|$ is

$$\begin{aligned}
\underline{0 < x \le 2} &: |z(x)|_{\text{dB}} \approx 0\\
\underline{x = 2} &: |z(x)|_{\text{dB}} = -3\text{dB} \quad \text{(this is the exact value)}\\
\underline{x > 2} &: |z(x)|_{\text{dB}} \approx -x \quad \text{(on log scale)}
\end{aligned}$$

which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 3 dB at $x = 2$ point (Fig. 7.5 (left, top)).

(d) *Phase limits*: Real and imaginary parts of an inverse complex function may be explicitly separated as

$$z(x) = \frac{1}{1+jx/2}\,\frac{1-jx/2}{1-jx/2} = \frac{1}{1+(x/2)^2} - j\frac{x/2}{1+(x/2)^2} \quad \therefore$$

$$\Re(z) = \frac{1}{1+(x/2)^2} \quad \text{and} \quad \Im(z) = \frac{-x/2}{1+(x/2)^2}.$$

Therefore, phase $\theta(z)$ limits are

$$\theta(z) = \arctan\left[\frac{\Im(z)}{\Re(z)}\right] = \arctan\left(\frac{-x/2}{1}\right) = -\arctan\left(\frac{x}{2}\right) \tag{7.14}$$

$$\therefore$$

$$\lim_{x\to 0}\theta(x) = \lim_{x\to 0}\left(-\arctan\left(\frac{x}{2}\right)\right) = -\arctan(0) = 0^\circ$$

$$\lim_{x\to 2}\theta(x) = \lim_{x\to 2}\left(-\arctan\left(\frac{x}{2}\right)\right) = -\arctan 1 = -\frac{\pi}{4} = -45^\circ$$

$$\lim_{x\to +\infty}\theta(x) = \lim_{x\to +\infty}\left(-\arctan\left(\frac{x}{2}\right)\right) = -\arctan(\infty) = -\frac{\pi}{2} = -90^\circ.$$

(e) *Phase plot*—Hand analysis of nonlinear function (7.14) is done by piecewise linear approximation as follows:
(a) $x \ll x_0$: $\theta(z) \approx -\arctan(0) = 0^\circ$.
(b) $x = 0.1x_0$: $\theta(z) = -\arctan(0.1) = -5.7^\circ \approx 0$.
(c) $x = x_0$: $\theta(z) = -\arctan 1 = -45^\circ$.
(d) $x = 10x_0$: $\theta(z) = -\arctan 10 = -84.3^\circ \approx 90^\circ$.
(e) $x \gg x_0$: $\theta(z) \approx -\arctan\infty = -90^\circ$
which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 5.7° at $x = x_0/10$ and $10\,x_0$ points (Fig. 7.5 (left, bottom)). Note that $\theta = -45^\circ$ is found at $x = x_0$.

7.10. Given $z(x) = 1/(1 + j^x/_{10})$, module and phase of a complex function that make Bode plot may be derived as follows:

(a) -3 dB *point*: By comparison with the general form (7.4), $x_0 = 10$.
(b) *Module limits*: This function is inverse complex; thus,

$$|z(x)| = \left|\frac{1}{1 + j\dfrac{x}{10}}\right| = \frac{1}{\sqrt{1 + \left(\dfrac{x}{10}\right)^2}} \tag{7.15}$$

$$\therefore$$

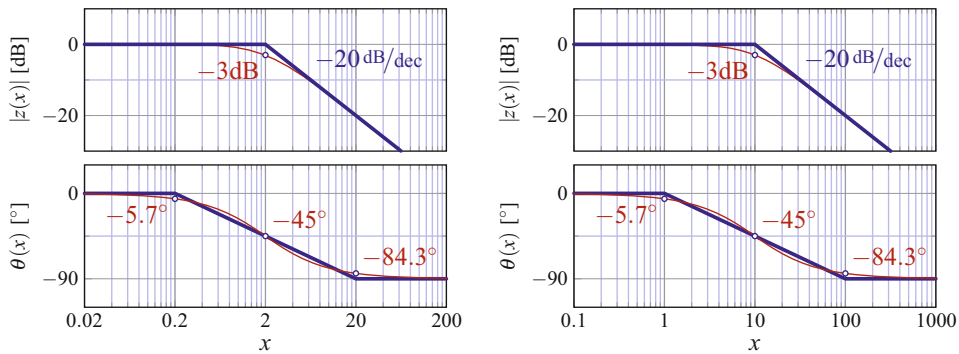


Fig. 7.5 Example P.7.9

$$\lim_{x\to 0}|z(x)| = \lim_{x\to 0}\frac{1}{\sqrt{1+\left(\frac{x}{10}\right)^2}} = 1 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 1 = 0\,\text{dB}$$

$$\lim_{x\to 10}|z(x)| = \lim_{x\to 10}\frac{1}{\sqrt{1+\left(\frac{x}{10}\right)^2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log\frac{1}{\sqrt{2}} = -3\,\text{dB}$$

$$\lim_{x\to+\infty}|z(x)| = \lim_{x\to+\infty}\frac{1}{\sqrt{1+\left(\frac{x}{10}\right)^2}} = 0 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 0 = -\infty.$$

(c) *Module (gain) plot*—Hand analysis of nonlinear function (7.15) is done by piecewise linear approximation as follows:

(a) $x \to 0$: $|z(x)|_{\text{dB}} = 0\,\text{dB}$.
(b) $x \ll x_0$: That is to say, $x/x_0 \to 0 \quad \Rightarrow \quad 1 + (x/x_0)^2 \approx 1$

$$|z(x)|_{\text{dB}} = 20\log\frac{1}{\sqrt{1+\left(\frac{x}{10}\right)^2}} \approx 20\log\frac{1}{\sqrt{1+0}} = \left\{\frac{1}{x} = x^{-1}, \quad \log a^b = b\log a\right\}$$
$$\approx -20\log 1 = 0\,\text{dB}.$$

(c) $x \gg x_0$: That is to say, $x/x_0 \gg 1 \quad \Rightarrow \quad 1 + (x/x_0)^2 \approx (x/x_0)^2$

$$|z(x)|_{\text{dB}} = 20\log\frac{1}{\sqrt{1+\left(\frac{x}{x_0}\right)^2}} \approx 20\log\frac{1}{\sqrt{\left(\frac{x}{x_0}\right)^2}} \approx 20\log\left|\frac{x_0}{x}\right|$$
$$= \left\{\log\frac{a}{b} = \log a - \log b\right\} = \underbrace{20\log x_0}_{\text{const.}} - 20\log|x|.$$

In conclusion, for $x \gg x_0$, module $|z(x)|_{\text{dB}}$ function asymptotically follows function $-x$ (on log scale) that crosses horizontal axis at $x = x_0$.

Therefore, piecewise linear approximation of $|z(x)| = |1/(1 + jx/10)|$ is

$$\begin{aligned} 0 < x \le 10 \ &: |z(x)|_{\text{dB}} \approx 0 \\ x = 10 \ &: |z(x)|_{\text{dB}} = -3\text{dB} \ \text{ (this is the exact value)} \\ x > 10 \ &: |z(x)|_{\text{dB}} \approx -x \ \text{ (on log scale)} \end{aligned}$$

which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 3 dB at $x = 10$ point (Fig. 7.5 (right, top)).

(d) *Phase limits*: Real and imaginary parts of an inverse complex function may be explicitly separated as

$$z(x) = \frac{1}{1 + jx/10}\,\frac{1 - jx/10}{1 - jx/10} = \frac{1}{1 + (x/10)^2} - j\frac{x/10}{1 + (x/10)^2} \quad \therefore$$

$$\Re(z) = \frac{1}{1 + (x/10)^2} \quad \text{and} \quad \Im(z) = \frac{-x/10}{1 + (x/10)^2}.$$

Therefore, as the real part of this inverse complex function is positive and its imaginary part is negative, phase $\theta(z)$ limits in the fourth quadrant are

$$\theta(z) = \arctan\left[\frac{\Im(z)}{\Re(z)}\right] = \operatorname{atan2}\left(\frac{-x/10}{1 + (x/10)^2}, \frac{1}{1 + (x/10)^2}\right) = -\arctan\left(\frac{x}{10}\right) \tag{7.16}$$

$$\therefore$$

$$\lim_{x\to 0}\theta(x) = \lim_{x\to 0}\left(-\arctan\left(\frac{x}{10}\right)\right) = -\arctan(0) = 0°$$

$$\lim_{x\to 10}\theta(x) = \lim_{x\to 10}\left(-\arctan\left(\frac{x}{10}\right)\right) = -\arctan 1 = -\frac{\pi}{4} = -45°$$

$$\lim_{x\to +\infty}\theta(x) = \lim_{x\to +\infty}\left(-\arctan\left(\frac{x}{10}\right)\right) = -\arctan(\infty) = -\frac{\pi}{2} = -90°.$$

(e) *Phase plot*—Hand analysis of nonlinear function (7.16) is done by piecewise linear approximation as follows:
(a) $x \ll x_0$: $\theta(z) \approx -\arctan(0) = 0°$.
(b) $x = 0.1x_0$: $\theta(z) = -\arctan(0.1) = -5.7° \approx 0°$.
(c) $x = x_0$: $\theta(z) = -\arctan 1 = -45°$.
(d) $x = 10x_0$: $\theta(z) = -\arctan 10 = -84.3° \approx 90°$.
(e) $x \gg x_0$: $\theta(z) \approx -\arctan\infty = -90°$
which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 5.7° at $x = x_0/10$ and $10\,x_0$ points (Fig. 7.5 (right, bottom)). Note that $\theta = -45°$ is found at $x = x_0$.

7.11. Given $z(x) = 1/(1 - jx/2)$, module and phase of a complex function that make Bode plot may be derived as follows:

(a) -3 dB *point*: By comparison with the general form (7.4), $x_0 = 2$.
(b) *Module limits*: This function is inverse complex; thus,

$$|z(x)| = \left|\frac{1}{1 - j\frac{x}{2}}\right| = \frac{1}{\sqrt{1 + \left(\frac{x}{2}\right)^2}} \tag{7.17}$$

$$\therefore$$

$$\lim_{x\to 0}|z(x)| = \lim_{x\to 0}\frac{1}{\sqrt{1 + \left(\frac{x}{2}\right)^2}} = 1 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 1 = 0\,\text{dB}$$

$$\lim_{x\to 2}|z(x)| = \lim_{x\to 2}\frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log\frac{1}{\sqrt{2}} = -3\,\text{dB}$$

$$\lim_{x\to+\infty}|z(x)| = \lim_{x\to+\infty}\frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = 0 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 0 = -\infty.$$

(c) *Module plot*—Hand analysis of nonlinear function (7.17) is done by piecewise linear approximation as follows:

(a) $\underline{x \to 0}$: $|z(x)|_{\text{dB}} = 0\,\text{dB}$.
(b) $\underline{x \ll x_0}$: That is to say, $x/x_0 \to 0 \quad \Rightarrow \quad 1 + (x/x_0)^2 \approx 1$

$$|z(x)|_{\text{dB}} = 20\log\frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}} \approx 20\log\frac{1}{\sqrt{1+0}} = \left\{\frac{1}{x} = x^{-1}, \quad \log a^b = b\log a\right\}$$

$$\approx -20\log 1 = 0\,\text{dB}.$$

(c) $\underline{x \gg x_0}$: That is to say, $x/x_0 \gg 1 \quad \Rightarrow \quad 1 + (x/x_0)^2 \approx (x/x_0)^2$

$$|z(x)|_{\text{dB}} = 20\log\frac{1}{\sqrt{1+\left(\frac{x}{x_0}\right)^2}} \approx 20\log\frac{1}{\sqrt{\left(\frac{x}{x_0}\right)^2}} \approx 20\log\left|\frac{x_0}{x}\right|$$

$$= \left\{\log\frac{a}{b} = \log a - \log b\right\} = \underbrace{20\log x_0}_{\text{const.}} - 20\log|x|.$$

In conclusion, for $x \gg x_0$, module $|z(x)|_{\text{dB}}$ function asymptotically follows function $-x$ (on log scale) that crosses horizontal axis at $x = x_0$.

Therefore, piecewise linear approximation of $|z(x)| = |1/(1 + jx/2)|$ is

$$\begin{aligned}
\underline{0 < x \le 2} &: |z(x)|_{\text{dB}} \approx 0\,\text{dB} \\
\underline{x = 2} &: |z(x)|_{\text{dB}} = -3\text{dB} \quad \text{(this is the exact value)} \\
\underline{x > 2} &: |z(x)|_{\text{dB}} \approx -x \quad \text{(on log scale)}
\end{aligned}$$

which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 3 dB at $x = 2$ point (Fig. 7.6 (left, top)).

(d) *Phase limits*: Real and imaginary parts of an inverse complex function may be explicitly separated as

$$z(x) = \frac{1}{1 - jx/2}\,\frac{1 + jx/2}{1 + jx/2} = \frac{1}{1 + (x/2)^2} + j\frac{x/2}{1 + (x/2)^2} \quad \therefore$$

$$\Re(z) = \frac{1}{1 + (x/2)^2} \quad \text{and} \quad \Im(z) = \frac{x/2}{1 + (x/2)^2}.$$

Therefore, as both real and imaginary parts of this inverse complex function are positive, phase $\theta(z)$ limits in the first quadrant are

$$\theta(z) = \arctan\left[\frac{\Im(z)}{\Re(z)}\right] = \operatorname{atan2}\left(\frac{x/2}{1+(x/2)^2}, \frac{1}{1+(x/2)^2}\right) = \arctan\left(\frac{x}{2}\right) \tag{7.18}$$

$$\therefore$$

$$\lim_{x\to 0}\theta(x) = \lim_{x\to 0}\arctan\left(\frac{x}{2}\right) = \arctan(0) = 0^\circ$$

$$\lim_{x\to 2}\theta(x) = \lim_{x\to 2}\arctan\left(\frac{x}{2}\right) = \arctan 1 = \frac{\pi}{4} = 45^\circ$$

$$\lim_{x\to +\infty}\theta(x) = \lim_{x\to +\infty}\arctan\left(\frac{x}{2}\right) = \arctan(\infty) = \frac{\pi}{2} = 90^\circ.$$

(e) *Phase plot*—Hand analysis of nonlinear function (7.18) is done by piecewise linear approximation as follows:
(a) $x \ll x_0$: . $\theta(z) \approx \arctan(0) = 0^\circ$
(b) $x = 0.1x_0$: $\theta(z) = \arctan(0.1) = 5.7^\circ \approx 0^\circ$.
(c) $x = x_0$: $\theta(z) = \arctan 1 = 45^\circ$.
(d) $x = 10x_0$: $\theta(z) = \arctan 10 = 84.3^\circ \approx 90^\circ$.
(e) $x \gg x_0$: $\theta(z) \approx \arctan\infty = 90^\circ$
which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 5.7° at $x = x_0/10$ and $10\,x_0$ points (Fig. 7.6 (left, bottom)). Note that $\theta = 45^\circ$ is found at $x = x_0$.

7.12. Given $z(x) = 1/(1 - j^x/_{10})$, module and phase of a complex function that make Bode plot may be derived as follows:

(a) -3 dB *point*: By comparison with the general form (7.4), $x_0 = 10$.
(b) *Module limits*: This function is inverse complex; thus,

$$|z(x)| = \left|\frac{1}{1 - j\dfrac{x}{10}}\right| = \frac{1}{\sqrt{1+\left(\dfrac{x}{10}\right)^2}} \tag{7.19}$$

$$\therefore$$

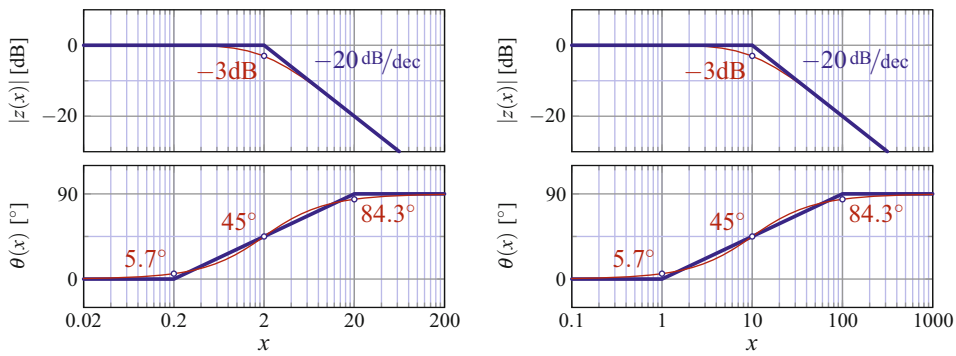


Fig. 7.6 Example P.7.11

$$\lim_{x\to 0}|z(x)| = \lim_{x\to 0}\frac{1}{\sqrt{1+\left(\frac{x}{10}\right)^2}} = 1 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 1 = 0\,\text{dB}$$

$$\lim_{x\to 10}|z(x)| = \lim_{x\to 10}\frac{1}{\sqrt{1+\left(\frac{x}{10}\right)^2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log\frac{1}{\sqrt{2}} = -3$$

$$\lim_{x\to +\infty}|z(x)| = \lim_{x\to +\infty}\frac{1}{\sqrt{1+\left(\frac{x}{10}\right)^2}} = 0 \quad \Rightarrow \quad |z(x)|_{\text{dB}} = 20\log 0 = -\infty.$$

(c) *Module plot*—Hand analysis of nonlinear function (7.19) is done by piecewise linear approximation as follows:

(a) $\underline{x \to 0}$: $|z(x)|_{\text{dB}} = 0\,\text{dB}$.
(b) $\underline{x \ll x_0}$: That is to say, $x/x_0 \to 0 \quad \Rightarrow \quad 1+(x/x_0)^2 \approx 1$

$$|z(x)|_{\text{dB}} = 20\log\frac{1}{\sqrt{1+\left(\frac{x}{10}\right)^2}} \approx 20\log\frac{1}{\sqrt{1+0}} = \left\{\frac{1}{x} = x^{-1}, \quad \log a^b = b\log a\right\}$$
$$\approx -20\log 1 = 0\,\text{dB}.$$

(c) $\underline{x \gg x_0}$: That is to say, $x/x_0 \gg 1 \quad \Rightarrow \quad 1+(x/x_0)^2 \approx (x/x_0)^2$

$$|z(x)|_{\text{dB}} = 20\log\frac{1}{\sqrt{1+\left(\frac{x}{x_0}\right)^2}} \approx 20\log\frac{1}{\sqrt{\left(\frac{x}{x_0}\right)^2}} \approx 20\log\left|\frac{x_0}{x}\right|$$
$$= \left\{\log\frac{a}{b} = \log a - \log b\right\} = \underbrace{20\log x_0}_{\text{const.}} - 20\log|x|.$$

In conclusion, for $x \gg x_0$, module $|z(x)|_{\text{dB}}$ function asymptotically follows function $-x$ (on log scale) that crosses horizontal axis at $x = x_0$.

Therefore, piecewise linear approximation of $|z(x)| = |1/(1+jx/10)|$ is

$$\underline{0 < x \le 10} \ : |z(x)|_{\text{dB}} \approx 0\,\text{dB}$$
$$\underline{x = 10} \ : |z(x)|_{\text{dB}} = -3\text{dB} \ \text{(this is the exact value)}$$
$$\underline{x > 10} \ : |z(x)|_{\text{dB}} \approx -x \ \text{(on log scale)}$$

which is to say as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 3 dB at $x = 10$ point (Fig. 7.6 (right, top)).

(d) *Phase limits*: Real and imaginary parts of an inverse complex function may be explicitly separated as

$$z(x) = \frac{1}{1 - j\,x/10}\,\frac{1 + j\,x/10}{1 + j\,x/10} = \frac{1}{1 + (x/10)^2} + j\,\frac{x/10}{1 + (x/10)^2} \quad \therefore$$

$$\Re(z) = \frac{1}{1 + (x/10)^2} \quad \text{and} \quad \Im(z) = \frac{x/10}{1 + (x/10)^2}.$$

Therefore, as both real and imaginary parts of this inverse complex function are positive, phase $\theta(z)$ limits in the first quadrant are

$$\theta(z) = \arctan\left[\frac{\Im(z)}{\Re(z)}\right] = \text{atan2}\left(\frac{x/10}{1 + (x/10)^2}, \frac{1}{1 + (x/10)^2}\right) = \arctan\left(\frac{x}{10}\right) \tag{7.20}$$

$\therefore$

$$\lim_{x\to 0} \theta(x) = \lim_{x\to 0}\left(\arctan\left(\frac{x}{10}\right)\right) = \arctan(0) = 0°$$

$$\lim_{x\to 10} \theta(x) = \lim_{x\to 10}\left(\arctan\left(\frac{x}{10}\right)\right) = \arctan 1 = \frac{\pi}{4} = 45°$$

$$\lim_{x\to +\infty} \theta(x) = \lim_{x\to +\infty}\left(\arctan\left(\frac{x}{10}\right)\right) = \arctan(\infty) = \frac{\pi}{2} = 90°.$$

(e) *Phase plot*—Hand analysis of nonlinear function (7.20) is done by piecewise linear approximation as follows:

(a) $x \ll x_0$: $\theta(z) \approx \arctan(0) = 0°$.
(b) $x = 0.1x_0$: $\theta(z) = \arctan(0.1) = 5.7° \approx 0$.
(c) $x = x_0$: $\theta(z) = \arctan 1 = 45°$.
(d) $x = 10x_0$: $\theta(z) = \arctan 10 = 84.3° \approx 90°$.
(e) $x \gg x_0$: $\theta(z) \approx \arctan\infty = 90°$

which is to say, as the function limits are the same for both exact and approximated functions, the largest error of piecewise linear approximation is 5.7° at $x = x_0/10$ and $10\,x_0$ points (Fig. 7.6 (right, bottom)). Note that $\theta = 45°$ is found at $x = x_0$.

7.2 Bode Plot Examples

7.13. This function is already factorized:

$$z(x) = (1 - jx)\left(1 + j\,\frac{x}{100}\right). \tag{7.21}$$

(a) Logarithmic form of factorized (7.21) is then

$$20\log z(x) = 20\log\left[\left(1 - j\,\frac{x}{1}\right)\left(1 + j\,\frac{x}{100}\right)\right] = \underbrace{+20\log\left(1 + j\,\frac{x}{1}\right)}_{\text{(2)}\ x_0=1} \underbrace{+20\log\left(1 + j\,\frac{x}{100}\right)}_{\text{(3)}\ x_0=100} \tag{7.22}$$

so that its first zero is at $x_0 = 1$ and its second zero at $x_0 = 100$. Note that separation between the two zeros is two decades.

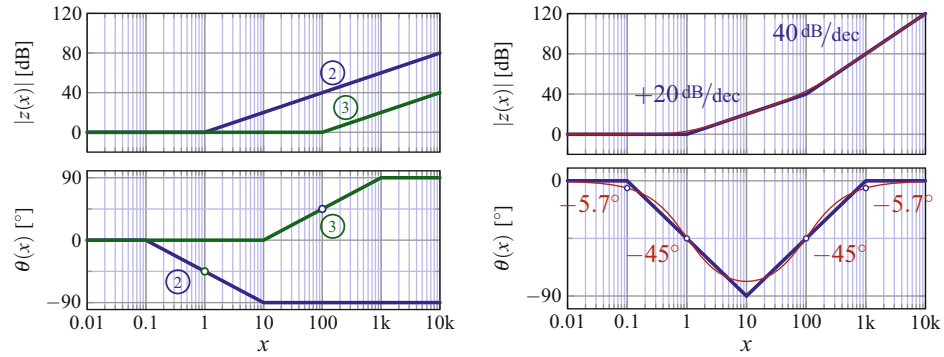


Fig. 7.7 Example P.7.13

(b) Evidently, both summation terms in (7.22) are in the basic form (7.3) (see P.7.1 to P.7.12). A simple addition of linear segments in Fig. 7.7 (left, top) results in complete piecewise linear approximation in Fig. 7.7 (right, top). Note how, at $x = 100$, two +20 dB/decade terms are amounted to +40 dB/decade thereafter.

(c) Similarly, phase functions of each individual segment in (7.22) are

$$\theta(z) = \underbrace{-\arctan\frac{x}{1}}_{\textcircled{2}\ x_0=1} \underbrace{+\arctan\frac{x}{100}}_{\textcircled{3}\ x_0=100}$$

as shown in Fig. 7.7 (left, bottom). A simple segment-by-segment sum produces complete piecewise linear approximation in Fig. 7.7 (right, bottom). Due to the two-decade separation, negative phase shift of the first term can almost complete to −90°, and then the positive phase shift of the second term starts and finishes two decades after at 0°.

7.14. This function may be factorized to explicitly show its poles and zeros as follows:

$$z(x) = 500\,\frac{1+jx}{100-jx} = \cancel{500}\ 5\,\frac{\left(1+j\frac{x}{1}\right)}{\cancel{100}\left(1-j\frac{x}{100}\right)} = 5\,\frac{1+j\frac{x}{1}}{1-j\frac{x}{100}}. \tag{7.23}$$

Evidently, gain factor equals $a_0 = 5$, zero is at $x_0 = 1$, and pole is at $x_0 = 100$. Note the two-decade separation between zero and pole.

(a) Logarithmic form of factorized (7.23) is then

$$\begin{aligned} 20\log z(x) &= 20\log\left[5\,\frac{1+j\frac{x}{1}}{1-j\frac{x}{100}}\right] \\ &= \underbrace{+20\log 5}_{\textcircled{1}\ a_0=5} \underbrace{+20\log\left(1+j\,\frac{x}{1}\right)}_{\textcircled{2}\ x_0=1} \underbrace{-20\log\left(1-j\,\frac{x}{100}\right)}_{\textcircled{3}\ x_0=100}. \end{aligned} \tag{7.24}$$

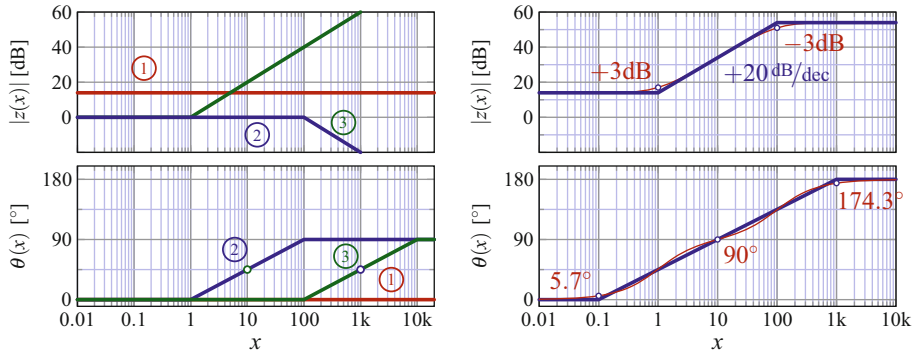


Fig. 7.8 Example P.7.14

(b) Evidently, each summation term in (7.24) is one of the basic forms in (7.1) to (7.4), and their individual piecewise linear approximations are calculated in P.7.1 to P.7.12. A simple addition of linear segments in Fig. 7.8 (left, top) results in complete piecewise linear approximation in Fig. 7.8 (right, top).

(c) Similarly, the sum of phase functions of each individual segment in (7.24) is

$$\theta(z) = \underbrace{\arctan 0}_{\textcircled{1}\ a_0=5} \underbrace{+\arctan\frac{x}{1}}_{\textcircled{2}\ x_0=1} \underbrace{-\arctan\frac{x}{100}}_{\textcircled{3}\ x_0=100}$$

shown in Fig. 7.8 (left, bottom). A simple segment-by-segment sum produces complete piecewise linear approximation in Fig. 7.8 (right, bottom). Note how each of two terms added phase shift of $+90°$: the numerator has positive imaginary term (thus $+90°$), while negative inverse term also adds $+90°$ totalling the phase shift to $+180°$ over four decades.

7.15. This function may be factorized to explicitly show its poles and zeros as follows:

$$z(x) = 100\,\frac{10+jx}{100+jx} = \cancel{100}\,\frac{10\left(1+j\frac{x}{10}\right)}{\cancel{100}\left(1+j\frac{x}{100}\right)} = 10\,\frac{1+j\frac{x}{10}}{1+j\frac{x}{100}} \tag{7.25}$$

where gain is $a_0 = 10$, zero is at $x_0 = 10$, and pole at $x_0 = 100$. Note the less than two-decade (i.e., only one decade) separation between pole and zero positions.

(a) Logarithmic form of factorized (7.25) is then

$$\begin{aligned} 20\log z(x) &= 20\log\left[10\,\frac{1+j\frac{x}{10}}{1+j\frac{x}{100}}\right] \\ &= \underbrace{+20\log 10}_{\textcircled{1}\ a_0=10} \underbrace{+20\log\left(1+j\,\frac{x}{10}\right)}_{\textcircled{2}\ x_0=10} \underbrace{-20\log\left(1+j\,\frac{x}{100}\right)}_{\textcircled{3}\ x_0=100}. \end{aligned} \tag{7.26}$$

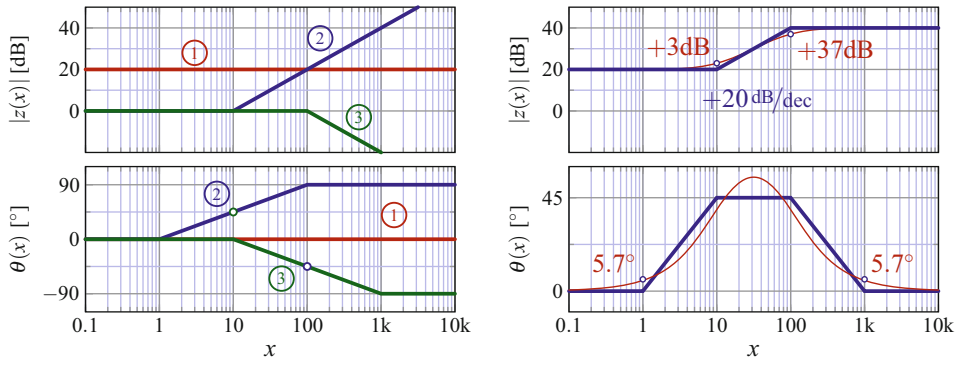


Fig. 7.9 Example P.7.15

(b) Evidently, summation terms in (7.26) are in basic forms in (7.1) and (7.3) (see P.7.1 to P.7.12). A simple addition of linear segments in Fig. 7.9 (left, top) results in complete piecewise linear approximation in Fig. 7.9 (right, top). Note that for arguments inferior to zero at $x_0 = 10$, the total gain is a_0, until pole at $x_0 = 100$ cancels the +20 dB/decade gain to settle the total gain at +40 dB.

(c) Similarly, the sum of phase functions of each individual segment in (7.26) is

$$\theta(z) = \underbrace{\arctan 0}_{\textcircled{1}\ a_0=10} \underbrace{+\arctan\frac{x}{10}}_{\textcircled{2}\ x_0=10} \underbrace{-\arctan\frac{x}{100}}_{\textcircled{3}\ x_0=100}$$

shown in Fig. 7.9 (left, bottom). A simple segment-by-segment sum produces complete piecewise linear approximation in Fig. 7.9 (right, bottom). Note that due to less than two-decade (i.e., only one decade) separation, positive phase shift of zero at $x_0 = 10$ is taken over by negative phase shift of pole at $x_0 = 100$. Consequently, there is a significant difference between the piecewise linear approximation (solid line) and the exact function (thin line).

7.16. This second-order function may be transformed into "standard form" that explicitly shows its poles and zeros as follows:

$$\begin{aligned}
H(j\omega) &= 2\,000\,\frac{2+jx}{220jx+4\,000-x^2} = \left\{(jx)^2=-x^2\right\} = 2\,000\,\frac{2+jx}{j^2x^2+220jx+4\,000} \\
&= 2\,000\,\frac{2+jx}{j^2x^2+20jx+200jx+4\,000} = 2\,000\,\frac{2+jx}{jx\,(20+jx)+200\,(20+jx)} \\
&= 2\,000\,\frac{2+jx}{(20+jx)(200+jx)} = \cancel{2\,000}\,\frac{\cancel{2}\left(1+j\frac{x}{2}\right)}{\cancel{20}\left(1+j\frac{x}{20}\right)\cancel{200}\left(1+j\frac{x}{200}\right)} \\
&= \frac{1+j\frac{x}{2}}{\left(1+j\frac{x}{20}\right)\left(1+j\frac{x}{200}\right)}
\end{aligned} \tag{7.27}$$

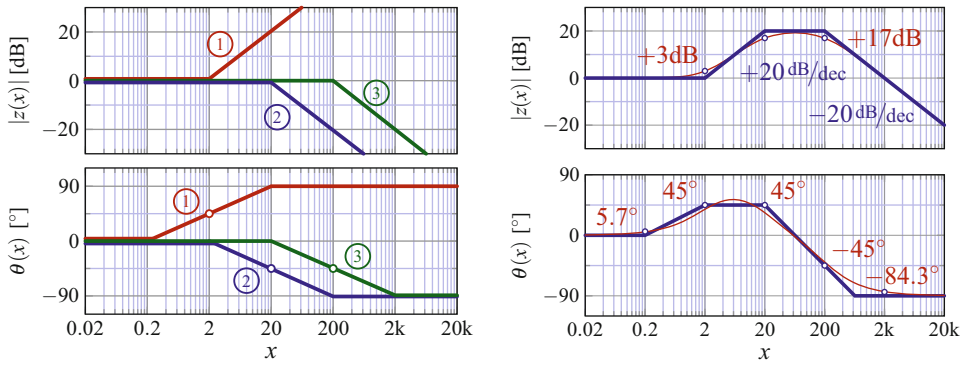


Fig. 7.10 Example P.7.16

where a single zero is at $x_0 = 2$, first pole at $x_0 = 20$, and the second pole at $x_0 = 200$. Note only one-decade separation between zero-pole-pole locations.

(a) Logarithmic form of factorized (7.27) is then

$$20\log z(x) = 20\log\left[\frac{1+j\,\frac{x}{2}}{\left(1+j\,\frac{x}{20}\right)\left(1+j\,\frac{x}{200}\right)}\right]$$

$$= \underbrace{+20\log\left(1+j\,\frac{x}{2}\right)}_{\textcircled{1}\ x_0=2}\ \underbrace{-20\log\left(1+j\,\frac{x}{20}\right)}_{\textcircled{2}\ x_0=20}\ \underbrace{-20\log\left(1+j\,\frac{x}{200}\right)}_{\textcircled{3}\ x_0=200}. \tag{7.28}$$

(b) Evidently, summation terms in (7.28) are in the basic forms (7.3) (see P.7.1 to P.7.12). A simple addition of linear segments in Fig. 7.10 (left, top) results in complete piecewise linear approximation in Fig. 7.10 (right, top).

(c) Similarly, the sum of phase functions of each individual segment in (7.28) is

$$\theta(z) = \underbrace{+\arctan\frac{x}{2}}_{\textcircled{1}\ x_0=2}\ \underbrace{-\arctan\frac{x}{20}}_{\textcircled{2}\ x_0=20}\ \underbrace{-\arctan\frac{x}{200}}_{\textcircled{3}\ x_0=200}$$

shown in Fig. 7.10 (left, bottom). A simple segment-by-segment sum produces complete piecewise linear approximation in Fig. 7.10 (right, bottom).

Linear Algebra

8

Problems

Exercises in this chapter are grouped into nine sections. Vector definitions and operations are followed by examples of linear transformations, determinants, and Cramer's rule. Then, modern interpretation of vectors and space is given in the form of matrix transformations, eigenvalues, and eigenvectors. Finally, linear algebra exercises are summarized by using one of the many methods to calculate the inverse matrix and powers of diagonalizable matrices.

8.1 Vector Definitions

Problems in P.8.1 to P.8.5 are a short review of vector definitions.

8.1. By a simple sketch, illustrate the geometrical interpretation of 3D, 2D, 1D, and 0D spaces, where "xD" refers to the number of dimensions.

8.2. What are the "preferred" forms of vector representations in physics, mathematics, and informatics?

8.3. Given three basis vectors (i.e., orthogonal) $\vec{i}, \vec{j}, \vec{k}$, sketch these three vectors in a single graph. Comment on the space defined by these three basis vectors.

$$\vec{i} = \begin{bmatrix} i_x \\ i_y \\ i_z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{j} = \begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

8.4. Given four points in 2D space: $A = (-2, 2)$, $B = (1, 4)$, $C = (5, 6)$, and $D = (3, 1)$,

(a) Sketch a graph showing $\overrightarrow{AB}$ and $\overrightarrow{CD}$ vectors.
(b) Write the matrix form of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ vectors.

8.5. Sketch a graph that shows the following vectors:

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023

R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41200-4_8

$$\vec{a} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

8.2 Vector Operations

Some of the basic vector calculations are in P.8.6 to P.8.8.

8.6. Calculate the magnitude of $\vec{p}$, that is to say, $|\vec{p}|$ if

(a) $\vec{p} = (0, 1)$ (b) $\vec{p} = (3, 4)$ (c) $\vec{p} = (-3, 1)$ (d) $\vec{p} = (2, 3, 6)$

8.7. Given vector

$$\vec{x} = \begin{bmatrix} x_x \\ x_y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

sketch the diagram of $\vec{a}, \vec{b}, \vec{c}$, if

(a) $\vec{a} = 2\,\vec{x}$ (b) $\vec{b} = \frac{2}{3}\,\vec{x}$ (c) $\vec{c} = -\frac{3}{4}\,\vec{x}$

8.8. Sketch a graph to illustrate the additions in 1D and 2D spaces. As an example, show the following vector additions:

(a) $2 + 3 = 5$ and $2 - 3 = -1$ (b) $\vec{c} = \vec{a} + \vec{b}$ if $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Given data in P.8.9 to P.8.13, while using the matrix form of vectors, calculate the final vector coordinates.

8.9. Given coordinates of points A and B in 2D space, calculate the coordinates of $\overrightarrow{AB}$ and $\overrightarrow{BA}$, then calculate $|\overrightarrow{AB}|$ and $|\overrightarrow{BA}|$, and note that $O = (0, 0)$.

(a) $A(4, 1)$, $B(1, -3)$
(b) $A(2, 3)$, $B(-1, 4)$
(c) $A(-1, -3)$, $B(4, 2)$
(d) $A(1, -2)$, $B(3, 2)$

8.10. Given coordinates of points A and B in 3D space, calculate the coordinates of $\overrightarrow{AB}$ and $\overrightarrow{BA}$, then calculate $|\overrightarrow{AB}|$ and $|\overrightarrow{BA}|$, and note that $O = (0, 0, 0)$.

(a) $A(4, 1, 6)$, $B(2, 4, -2)$
(b) $\overrightarrow{OA} = (2, 3, 4)$, $\overrightarrow{OB} = (3, 0, -1)$

8.11. Given $\vec{a} = (2, -3), \vec{b} = (3, -1)$, calculate $\vec{c} = \vec{a} + \vec{b}$, $\vec{d} = \vec{a} - \vec{b}$.

8.12. Given $A(2, 3, 1)$, $B(3, -1, 0)$, $C = (-1, -2, 1)$, $D = (-3, 3, -2)$, calculate:

(a) $\vec{m} = \overrightarrow{AB} + \overrightarrow{CD}$ (b) $\vec{n} = \overrightarrow{AB} - \overrightarrow{CD}$

8.13. Given $\vec{a} = (-1, 2)$, $\vec{b} = (3, 2)$, $\vec{c} = (-2, -3)$, calculate:

(a) $\vec{p} = \vec{a} + \vec{b}$, (b) $\vec{m} = 2\vec{a} - \vec{b}$, (c) $\vec{n} = 3\vec{a} - \vec{b} - \vec{c}$, (d) $\vec{r} = 2\vec{b} - 1/2\,\vec{c}$

8.14. Vectors $\vec{a}$ and $\vec{b}$ create angle $\theta = \pi/6$. Given that $|\vec{a}| = \sqrt{3}$ and $|\vec{b}| = 1$, calculate angle α between vectors $\vec{p} = \vec{a} + \vec{b}$ et $\vec{q} = \vec{a} - \vec{b}$.

8.3 Linear Transformations

Two or more vectors are said to be linearly independent if none of them can be written as a linear combination of the others.

Reminder: Linear combination of two vectors is written as

$$\vec{z} = a\vec{x} + b\vec{y} = a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{cases} ax_1 + by_1 \\ ax_2 + by_2 \end{cases}$$

where (a, b) are coefficients, (x_1, x_2) are coordinates of $\vec{x}$, and (y_1, y_2) are coordinates of $\vec{y}$.

8.15. Explain briefly the similarities and differences between a *function* $f(x)$ and a *linear transformation* $L(\vec{v})$ operations. What is the relationship between linear transformations and space?

8.16. Given, as an example, vector $\vec{v} = -2\vec{i} + \vec{j}$, with a simple diagram:

(a) Show that the numerical form of $\vec{v} = -2\vec{i} + \vec{j}$ stays the same after an arbitrary linear transformation $L(\vec{v})$.
(b) Calculate the matrix form of $L(\vec{v})$ used in your example.

8.17. Show the geometrical interpretation of linear transformation $L(\vec{x})$ given that

$$L = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

8.18. With a simple 2D graph, illustrate each of the linear space transformations performed as:

(a) Rotation 90° counterclockwise
(b) Shear 45° clockwise
(c) First rotation 90° counterclockwise and then shear 45° clockwise

8.19. Given vectors $\vec{a} = (2, 1)$ and $\vec{b} = (1, 0)$, write $\vec{m} = (9, 1)$ as a linear combination of $\vec{a}$ and $\vec{b}$.

8.20. Given vectors $\vec{u} = (3, -1)$, $\vec{v} = (1, -2)$, and $\vec{w} = (-1, 7)$, write $\vec{m} = \vec{u} + \vec{v} + \vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$.

8.4 Determinants

Given two or more vectors in a matrix form, its determinant may be interpreted as a multiplication factor that measures the signed (i.e., positive or negative) size of the 2D surface (or multidimensional volume) enclosed by these vectors. Specifically, positive/negative sign of the determinant indicates the order in which the vectors are taken. As an analogy, "positive" and "negative" 2D surfaces may be associated with the two sides of a sheet of paper. That is to say, the two surfaces have same absolute value, but opposite orientations.

Reminder: From the vector graph, it can be easily shown that the second-order determinant of matrix A made of two vectors $\vec{x}$ and $\vec{y}$ is calculated as the difference between products along its two diagonals, positive along blue and negative along red direction, as

$$\det(A) = \Delta A = |A| = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = (x_1 \times y_2) - (x_2 \times y_1)$$

where $\det(A)$, or ΔA, or $|A|$ reads as the "determinant of matrix A" (not "the absolute value of A" or "delta A"). The determinant's value may be positive, negative, or zero.

In general, higher-order determinants may be calculated by two principal methods: by calculating cross-products or by the method of cofactor expansions.

Reminder: **Method 1:** by calculating cross-products. Following the same idea as for the second-order determinants, n-th-order determinants may be calculated by writing the extended determinant form, that is to say, to repeat the first $(n-1)$ columns on the right side of its $n \times n$ determinant. By doing so, it is possible to complete positive (blue) and negative (red) diagonals and then to calculate all cross-products as (e.g., as in 3×3 determinant)

$$\Delta_3 = \begin{vmatrix} x_1 & y_1 & z_1 & x_1 & y_1 \\ x_2 & y_2 & z_2 & x_2 & y_2 \\ x_3 & y_3 & z_3 & x_3 & y_3 \end{vmatrix}$$

$$\begin{aligned} &= +(x_1 \times y_2 \times z_3) + (y_1 \times z_2 \times x_3) + (z_1 \times x_2 \times y_3) \\ &\quad -(x_3 \times y_2 \times z_1) - (y_3 \times z_2 \times x_1) - (z_3 \times x_2 \times y_1) \end{aligned}$$

Reminder: **Method 2:** by cofactor expansions along the i-th row of the n-th-order determinant A. Given matrix A as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \therefore \quad |A| = \sum_{i=1}^{k} (-1)^{i+j} a_{ij} \, |A_{i,j}|$$

(continued)

(continued)
where the $(i,\ j)$ *minor* that is usually denoted as $A_{i,j}$ is the $(n-1)\times(n-1)$ matrix obtained from A by deleting the i-th row and the j-th column while keeping their common term a_{ij}. Note that, by systematically reducing higher-order determinants until there is only a linear combination of second-order determinants, it is possible to calculate the total determinant. In addition, any row or column may be chosen for the expansion.

Calculate the determinants given in P.8.21 to P.8.34.

8.21. $\begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}$ **8.22.** $\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$ **8.23.** $\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix}$ **8.24.** $\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix}$

8.25. $\begin{vmatrix} \sqrt{3} & -3\sqrt{2} \\ \sqrt{3} & 2\sqrt{2} \end{vmatrix}$ **8.26.** $\begin{vmatrix} 1 & i \\ -i & 1 \end{vmatrix}$ **8.27.** $\begin{vmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{vmatrix}$

8.28. $\begin{vmatrix} 2 & 3 & 4 \\ 5 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix}$ **8.29.** $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix}$ **8.30.** $\begin{vmatrix} 1 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 1 \end{vmatrix}$

8.31. $\begin{vmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ 8 & 9 & 7 \end{vmatrix}$ **8.32.** $\begin{vmatrix} 2 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 1 & -3 \end{vmatrix}$ **8.33.** $\begin{vmatrix} 5 & 0 & 4 \\ 8 & 0 & -7 \\ 3 & 2 & 1 \end{vmatrix}$ **8.34.** $\begin{vmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{vmatrix}$

What is the geometrical interpretation of matrices given P.8.35 to P.8.37?

8.35. $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ **8.36.** $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ **8.37.** $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

8.5 Cramer's Rule

Solve the second-order systems of linear equations in P.8.38 to P.8.41.

8.38. $\begin{aligned} 4x_1 - 3x_2 &= 0 \\ 2x_1 + 3x_2 &= 18 \end{aligned}$ **8.39.** $\begin{aligned} x + 2y &= -5 \\ 3x - y &= 13 \end{aligned}$ **8.40.** $\begin{aligned} x + 2y &= -8 \\ 2x - y &= -1 \end{aligned}$ **8.41.** $\begin{aligned} a + b &= 7 \\ 2a + 2b &= 14 \end{aligned}$

Solve the third-order systems of linear equations in P.8.42 to P.8.47.

8.42. $\begin{aligned} 5x - 5y - 15z &= 40 \\ 4x - 2y - 6z &= 19 \\ 3x - 6y - 17z &= 41 \end{aligned}$ **8.43.** $\begin{aligned} 2x - 3y - z &= 5 \\ x + y + 2z &= 7 \\ 2x - y - z &= 1 \end{aligned}$ **8.44.** $\begin{aligned} x + y + z &= 3 \\ x - y + z &= 5 \\ -x + y - z &= 10 \end{aligned}$

8.45. $x + y + z = 36$
$2x - z = -17$
$6x - 5z = 7$

8.46. $x + y = 7$
$y + z = 8$
$-x + 2z = 7$

8.47. $5x - 13y + 13z = 8$
$x - 3y + z = 4$
$-x + 2y - 5z = 3$

Problems of curve fitting are translated into the problem of solving system of equations, P.8.48 to P.8.49.

8.48. By using the curve fitting method, derive a linear function $f(x) = ax + b$ that best fits given coordinates of two points $A = (-2, 20)$and $B = (1, 5)$ in 2D space. Can you fit quadratic function $g(x) = ax^2 + bx + c$ for the same two data points? Any other nonlinear function?

8.49. By using the curve fitting method, derive a quadratic function $f(x) = ax^2 + bx + c$ that best fits given coordinates of three data points $A = (-2, 20)$, $B = (1, 5)$, and $C = (3, 25)$ in 2D space. Can you fit a fourth-order polynomial function for the same three data points? Any other nonlinear function?

In P.8.50 to P.8.51, given $\vec{a}$, $\vec{b}$, and $\vec{c}$, show that they are linearly independent and then write $\vec{c}$ as a linear combination of $\vec{a}$ and $\vec{b}$.

8.50. $\vec{a} = (3, -2)$, $\vec{b} = (-2, 1)$, $\vec{c} = (7, -4)$

8.51. $\vec{a} = (1, 2)$, $\vec{b} = (3, 4)$, $\vec{c} = (0, -2)$

In P.8.52 to P.8.53, calculate parameter λ so that given vectors are linearly independent:

8.52. $\vec{a} = (3, \lambda)$, $\vec{b} = (2, 6)$

8.53. $\vec{a} = (6, 8, 4)$, $\vec{b} = (3, 4, 2)$, $\vec{c} = (\lambda, 0, 1)$

8.6 Vector Space

In P.8.54 to P.8.59, calculate the matrix–vector products, and show its geometric interpretation.

8.54. $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

8.55. $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1+\sqrt{3} \\ 2 \end{bmatrix}$

8.56. $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1-\sqrt{3} \\ 2 \end{bmatrix}$

8.57. $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

8.58. $\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

8.59. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

In P.8.60 to P.8.65, calculate the matrix–matrix products, and show its geometric interpretation.

8.60. $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

8.61. $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

8.62. $\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

8.63. $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

8.64. $\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

8.65. $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

In P.8.66 to P.8.69, calculate the products of non-square matrices.

8.66. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

8.67. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

8.68. $\begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 0 & 1 \end{bmatrix}$

8.69. $\begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} 3 & 0 & 5 \\ -2 & -1 & 4 \end{bmatrix}$

8.7 Eigenvalues and Eigenvectors

The geometrical interoperation of a matrix is that it describes the transformation of the space itself and by consequence all objects within. In some cases, however, there are vectors that do not change their direction before and after the transformation. If existing, these vectors are very useful in many engineering applications.

Reminder: Eigenvectors may be described as vectors that do not change their direction after some space transformation. By definition, if the following equation is satisfied:

$$A\ \vec{v} = \lambda\ \vec{v}$$

then $\lambda \in \mathbb{C}$ is a constant referred to as *eigenvalue*, and $\vec{v}$ is its associated *eigenvector* of matrix A.

Furthermore, the eigenvalues and eigenvectors definition equation may be rewritten as

$$A\,\vec{v} = \lambda\,\vec{v} \quad \Rightarrow \quad A\,\vec{v} - \lambda\,\vec{v} = \vec{0} \quad \Rightarrow \quad (A - \lambda\, I)\ \vec{v} = \vec{0}$$

where I is the *identity matrix* (i.e., a square matrix of the same size as A with ones on the main diagonal and zeros elsewhere). In order to satisfy this equation, as $\vec{v} \neq 0$, it must be that the *determinant* of $(A - \lambda\, I)$ equals zero; thus,

$$\det\,(A - \lambda\, I) = 0$$

is the formal definition of *the characteristic polynomial* of matrix A.

In P.8.70 to P.8.71, derive $\vec{v}$ and its associated eigenvalue λ if existing.

8.70. $A = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

8.71. $A = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Given matrices in P.8.72 to P.8.76, calculate their respective eigenvalues λ and eigenvectors $\vec{v}$, if existing.

8.72. $\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$ **8.73.** $\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$ **8.74.** $\begin{bmatrix} 10 & -5 \\ 5 & 2 \end{bmatrix}$ **8.75.** $\begin{bmatrix} 2 & -5 \\ 4 & 6 \end{bmatrix}$

8.76. $\begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix}$

In P.8.77 to P.8.78, prove that $\vec{v}$ is an eigenvector of its associated matrix A, and then determine its eigenvalue λ.

8.77. $A = \begin{bmatrix} -2 & -9 & -1 \\ 2 & -6 & -5 \\ -4 & -3 & 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ **8.78.** $A = \begin{bmatrix} -2 & -9 & -1 \\ 2 & -6 & -5 \\ -4 & -3 & 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$

Given matrix in P.8.79 to P.8.82, calculate eigenvalue(s) λ and eigenvector(s), if existing.

8.79. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 4 & 2 \end{bmatrix}$ **8.80.** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{bmatrix}$ **8.81.** $\begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix}$

8.82. $\begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

Calculate eigenvalues λ and eigenvectors $\vec{v}$ for matrices in P.8.83 to P.8.88. Note that eigenvalues of these matrices have "algebraic multiplicity" (geometric multiplicity is left for another time).

8.83. $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ **8.84.** $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ **8.85.** $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

8.86. $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ **8.87.** $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ **8.88.** $\begin{bmatrix} 1 & -2 & -1 \\ 1 & 7 & 1 \\ -2 & -4 & 3 \end{bmatrix}$

8.8 Matrix Inversion

Similar to "regular numbers" where a number multiplied by its inverse is equal to one, matrices may have their inverse, so that their product is equal to "unitary matrix".

Reminder: Inverse A^{-1} of a matrix A exists if its determinant does not equal to zero, i.e.,

$$A\,A^{-1} = A^{-1}\,A = I, \quad \text{if} \quad \Delta_A \neq 0$$

where I is the unitary matrix. Note that the multiplication of matrix and inverse is commutative.

Given matrices in P.8.89 to P.8.91, calculate their respective inverse matrices.

8.89. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

8.90. $\begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix}$

8.91. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

8.9 Powers of Diagonalizable Matrices

Examples in P.8.92 to P.8.97 are not to be solved by "brute force" matrix multiplication; instead, the goal is to master one of the general procedures for calculating matrix powers.

8.92. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^2$

8.93. $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^5$

8.94. $\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}^5$

8.95. $\begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}^3$

8.96. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}^4$

8.97. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{10}$

Answers

8.1 Vector Definitions

8.1. The geometrical interpretation of space is based on the definition of a *point*, which is visualized as a sphere whose radius equals *zero*, Fig. 8.1. Thus, assuming nothing else exists, a point would represent "0-order space" annotated as R^0.

(a) *Line:* Multiple points aligned next to each other form *line*. Therefore, it takes *infinitely* points to create a line of any *finite* length L. If nothing else exists, a line represents *one-dimensional* space, annotated as R^1. The position of any point relative to the one that is chosen to be the *origin* (thus numbered as "0") is associated with a unique number showing the distance from the origin, which is in either positive ("+") or negative ("−") direction.
(b) *Surface:* It takes infinitely lines in parallel to create a finite two-dimensional *surface*, annotated as R^2. Analogy to R^2 space would be a sheet of paper whose thickness equals zero. In this case, the position of any point making the surface relative to the origin is described by two numbers, which are referred to as "coordinates" and written as a two-number pair (x, y), one for "negative–positive" and one for "south–north" distance.
(c) *Volume:* A stack of surfaces creates a *volume*, annotated as R^3 space. Visual analogy would be a book where the position of any point is described by *three* numbers, (x, y, z), one for "negative–positive", one for "south–north", and one for "up–down" distance.

Although human visualization of space is limited to three dimensions, note that in mathematics, however, there is no limit on the order of space. Each dimension is described by one variable $x, y, z, \ldots$ (thus one equation). In linear algebra, a n-dimensional matrix may be seen as a compact form of writing a system of n equations, thus describing n-dimensional space. Note that each lower-order space can be seen as one of the possible projections ("shadows") of the higher-order space. For example, if looked from "straight ahead", a line looks like a point. Looked from "a side", a surface looks like a line. Looked from one direction, a cube (3D) appears as a square (2D), or a sphere (3D) casts a shadow like a circle (2D), etc.

8.2. A vector signifies a variable that has *two* properties: magnitude and direction. That is to say, a simple number is not a vector because it has only magnitude ("length") but not direction.

In physics, "vector" takes geometrical interpretation in graphical form of a "directed arrow" that indicates both magnitude (i.e., the arrow length) and direction (i.e., from its tail to the head).

In mathematics, a vector variable is annotated in algebraic form as

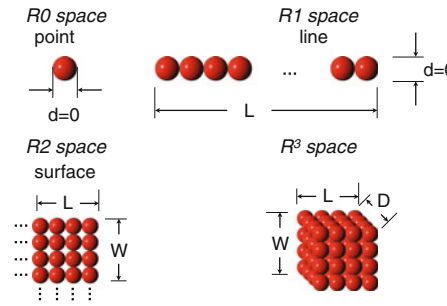


Fig. 8.1 Example P.8.1

$$\begin{aligned} \underline{1D:} \quad & \vec{a} = a_x\vec{i} \\ \underline{2D:} \quad & \vec{a} = a_x\vec{i} + a_y\vec{j} \\ \underline{3D:} \quad & \vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \\ & \text{etc.} \end{aligned}$$

where (a_x, a_y, a_z) are vector magnitudes as measured along the three directions $(\vec{i}, \vec{j}, \vec{k})$, respectively (also referred to as the "unity vectors").

In informatics, the preferred way of representing a vector is in one-column matrix form, as, for example,

$$\vec{a} = \begin{bmatrix} 3.14 \\ -5 \\ 2 \end{bmatrix}$$

where the column list may be interpreted as a list of orthonormal (i.e., orthogonal and normalized) coordinates in each direction relative to the origin (0, 0, 0), in this case 3D space.

8.3. As defined, the three unity vectors are used to define 3D space and are written in a form of 3×3 matrix as

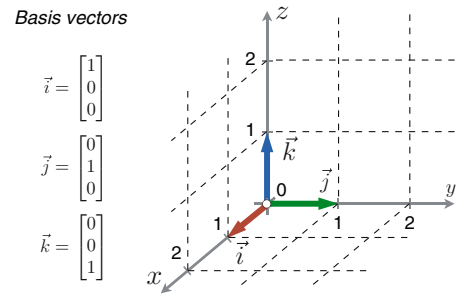
$$(\vec{i}, \vec{j}, \vec{k}) = \begin{bmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 1 \end{bmatrix}$$

where the first column is for $\vec{i}$, the second column is for $\vec{j}$, and the third column is for $\vec{k}$ unity vector, Fig. 8.2. The rows are then coordinates in each of the three normal directions, here (x, y, z). This resulting matrix where all elements along the main diagonal equal one and all other matrix elements equal zero is known as "unity matrix" with properties similar to the number one.

8.4. By convention, vectors are oriented in direction from the first to the second coordinate.

(a) Given four points in 2D space, $A = (-2, 2)$, $B = (1, 4)$, $C = (5, 6)$, and $D = (3, 1)$, vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are as shown in Fig. 8.3 (left).
(b) By writing the matrix form, it is assumed that all vectors originate at (0, 0) point, i.e., that stating only the end coordinates is sufficient. Geometrically, that means vectors are translated so that

Fig. 8.2 Example P.8.3



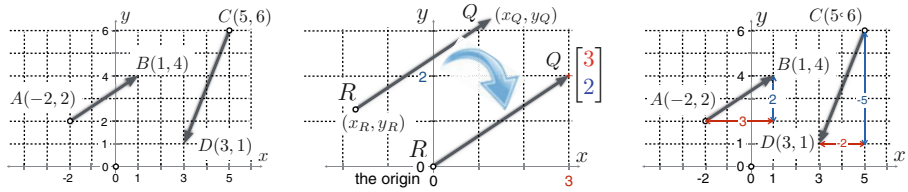
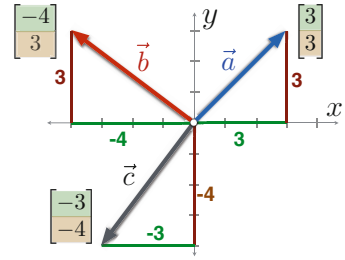


Fig. 8.3 Example P.8.4

Fig. 8.4 Example P.8.5



their origin is at (0, 0) point, for example, as the final position of $\vec{RQ}$; see Fig. 8.3 (center).

$$\overrightarrow{RQ} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Similarly, after being translated to the origin (see Fig. 8.3 (right)), vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are written as

$$\overrightarrow{AB} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \overrightarrow{CD} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

where "3" is the horizontal projection of $\overrightarrow{AB}$, "2" is the vertical projection of $\overrightarrow{AB}$, and they are both positive as oriented relative to the vector origin at $(-2, 2)$. And, "-2" is the horizontal projection of $\overrightarrow{CD}$, "-5" is the vertical projection of $\overrightarrow{AB}$, and they are both negative as oriented relative to the vector origin at $(5, 6)$. By definition, translated vectors are still identical: they have the same magnitude and direction.

8.5. Given

$$\vec{a} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

By convention, the vector matrix form lists coordinates (x, y) in the vertical order and relative to the origin $(0, 0)$; therefore, the three vectors are as in Fig. 8.4.

8.2 Vector Operations

8.6. By Pythagoras' theorem, given p_x, p_y, and p_z vector projections, it follows that

(a) $\vec{p} = (0, 1) \;\therefore\; |\vec{p}| = \sqrt{0^2 + 1^2} = 1$

(b) $\vec{p} = (3, 4) \;\therefore\; |\vec{p}| = \sqrt{3^2 + 4^2} = 5$

(c) $\vec{p} = (-3, 1) \;\therefore\; |\vec{p}| = \sqrt{3^2 + 1^2} = \sqrt{10}$

(d) $\vec{p} = (2, 3, 6) \;\therefore\; |\vec{p}| = \sqrt{2^2 + 3^2 + 6^2} = 7$

8.7. Multiplying the vector by a constant does not change the vector's direction, only its magnitude; see Fig. 8.5. The constant multiplies all vector components, in 2D case both x_x and x_y.

(a) $\vec{a} = 2\,\vec{x} = 2 \begin{bmatrix} x_x \\ x_y \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \times 3 \\ 2 \times 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

(b) $\vec{b} = \frac{2}{3}\,\vec{x} = \frac{2}{3} \begin{bmatrix} x_x \\ x_y \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \times 3 \\ 2/3 \times 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4/3 \end{bmatrix}$

(c) $\vec{c} = -\frac{3}{4}\,\vec{x} = -\frac{3}{4} \begin{bmatrix} x_x \\ x_y \end{bmatrix} = -\frac{3}{4} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/4 \times 3 \\ -3/4 \times 2 \end{bmatrix} = \begin{bmatrix} -9/4 \\ -3/2 \end{bmatrix}$

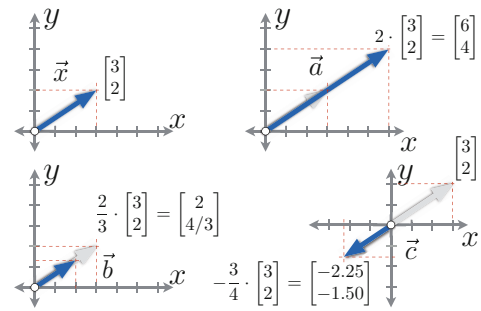
8.8. Geometrically, vector addition is done by a simple chaining of the vectors. The vector representing the total sum starts at the tail of the first and ends at the head of the last vector in this chain.

(a) Note that all vectors in 1D space are collinear (i.e., parallel, because there is only one direction possible); therefore, their direction is already known. As a consequence, their magnitudes are sufficient. In the more general sense, real and complex numbers are also seen as vectors relative in the origin. As illustrated in Fig. 8.6 (left), the sum of two numbers equals to the simple sum of their respective magnitudes.

$$2 + 3 = 5 \quad \text{and} \quad 2 + (-3) = -1$$

(b) In 2D space, not all vectors are collinear, which is to say that in general, the sum of two vectors is not equal to the simple sum of their magnitudes. Instead, the rules of triangles must be applied; see Fig. 8.6 (right). Note that in the matrix notification, it is the modules of vector projections that add in the same manner as "regular" numbers. For example, given that

Fig. 8.5 Example P.8.7


y
x
$\vec{x}$
$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$
$2 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$
$\vec{a}$
$\frac{2}{3} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4/3 \end{bmatrix}$
$\vec{b}$
$-\frac{3}{4} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2.25 \\ -1.50 \end{bmatrix}$
$\vec{c}$

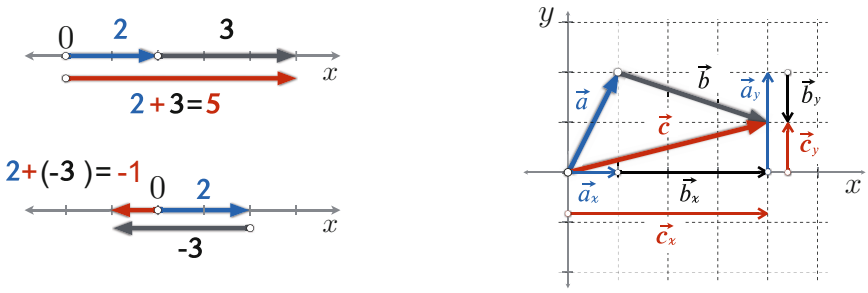


Fig. 8.6 Example P.8.8

$$\vec{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

it follows that

$$\vec{c} = \vec{a} + \vec{b} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+3 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} c_x \\ c_y \end{bmatrix}$$

8.9. Given $\overrightarrow{AB}$ and $\overrightarrow{BA} = -\overrightarrow{AB}$, their geometrical and matrix representations in 2D are as follows.

(a) Coordinates of $\overrightarrow{AB}$ are calculated as the end point minus the starting point, i.e.,

$$\overrightarrow{AB} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} (1-4) \\ (-3-1) \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

$$\overrightarrow{BA} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} (4-1) \\ (1-(-3)) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

By inspection of graph in Fig. 8.7, both $|\overrightarrow{AB}|$ and $|\overrightarrow{BA}|$ are hypotenuses of the same right-angled triangle, thus calculated by Pythagoras' theorem as

$$|\overrightarrow{AB}| = |\overrightarrow{BA}| = \sqrt{|x_B - x_A|^2 + |y_B - y_A|^2} = \sqrt{3^2 + 4^2} = 5$$

(b) Similarly,

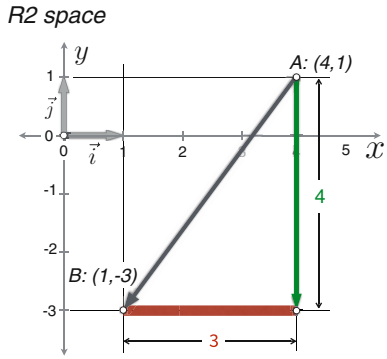
$$\overrightarrow{AB} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (-1-2) \\ (4-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\overrightarrow{BA} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} (2-(-1)) \\ (3-4) \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Thus,

$$|\overrightarrow{AB}| = |\overrightarrow{BA}| = \sqrt{|x_B - x_A|^2 + |y_B - y_A|^2} = \sqrt{3^2 + 1^2} = 2$$

Fig. 8.7 Example P.8.9



(c) Likewise,

$$\overrightarrow{AB} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} (4-(-1)) \\ (2-(-3)) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\overrightarrow{BA} = \begin{bmatrix} -1 \\ -3 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} (-1-4) \\ (-3-2) \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$$

Thus,

$$|\overrightarrow{AB}| = |\overrightarrow{BA}| = \sqrt{|x_B - x_A|^2 + |y_B - y_A|^2} = \sqrt{5^2 + 5^2} = 5\sqrt{2}$$

(d) And,

$$\overrightarrow{AB} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} (3-1) \\ (2-(-2)) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\overrightarrow{BA} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} (1-3) \\ (-2-2) \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

so that

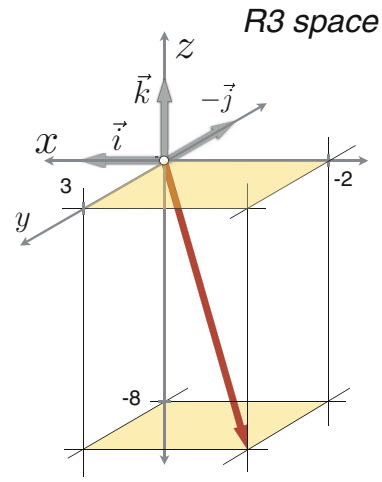
$$|\overrightarrow{AB}| = |\overrightarrow{BA}| = \sqrt{|x_B - x_A|^2 + |y_B - y_A|^2} = \sqrt{2^2 + 2^2} = 2$$

8.10. Given $\overrightarrow{AB}$ and $\overrightarrow{BA} = -\overrightarrow{AB}$, their geometrical and matrix representations in 3D are as follows.

(a) Coordinates of $\overrightarrow{AB}$ are calculated as the end point minus the starting point,

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} (2-4) \\ (4-1) \\ (-2-6) \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -8 \end{bmatrix}$$

Fig. 8.8 Example P.8.10



$$\overrightarrow{BA} = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} (4-2) \\ (1-4) \\ (6-(-2)) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$$

By inspection of graph in Fig. 8.8, both $|\overrightarrow{AB}|$ and $|\overrightarrow{BA}|$ are hypotenuses of the same right-angled triangle that is placed in the diagonal of a cuboid; thus, by Pythagoras' theorem,

$$|\overrightarrow{AB}| = |\overrightarrow{BA}| = \sqrt{|x_B - x_A|^2 + |y_B - y_A|^2 + |z_B - z_A|^2} = \sqrt{2^2 + 3^2 + 8^2} = \sqrt{77}$$

(b) Relative to the point of origin,

$$\overrightarrow{AB} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (3-2) \\ (0-3) \\ (-1-4) \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}$$

$$\overrightarrow{BA} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} (2-3) \\ (3-0) \\ (4-(-1)) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$$

By inspection of graph in Fig. 8.8, $\overrightarrow{AB}$ coincides with the diagonal of a cuboid, thus $|\overrightarrow{AB}|$ and $|\overrightarrow{BA}|$ as

$$|\overrightarrow{AB}| = |\overrightarrow{BA}| = \sqrt{|x_B - x_A|^2 + |y_B - y_A|^2 + |z_B - z_A|^2} = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}$$

8.11. Given that $\vec{a} = (2, -3)$ and $\vec{b} = (3, -1)$ are already placed at the origin,

$$\vec{c} = \vec{a} + \vec{b} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

$$\vec{c} = \vec{a} - \vec{b} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

8.12. Given $A(2, 3, 1)$, $B(3, -1, 0)$, $C = (-1, -2, 1)$, and$D = (-3, 3, -2)$, first, all vectors are translated to the origin as

$$\overrightarrow{AB} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} (3-2) \\ (-1-3) \\ (0-1) \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$$

$$\overrightarrow{CD} = \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} (-3-(-1)) \\ (3-(-2)) \\ (-2-1) \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ -3 \end{bmatrix}$$

Then,

$$(a)\ \ \vec{m} = \overrightarrow{AB} + \overrightarrow{CD} = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}$$

$$(b)\ \ \vec{n} = \overrightarrow{AB} - \overrightarrow{CD} = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 2 \end{bmatrix}$$

8.13. Similar to A.8.12,

(a) $\vec{p} = \vec{a} + \vec{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(b) $\vec{m} = 2\vec{a} - \vec{b} = 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} (2\times(-1)-3) \\ (2\times 2-2) \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$

(c) $\vec{n} = 3\vec{a} - \vec{b} - \vec{c} = 3\begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} (3\times(-1)-3-(-2)) \\ (3\times 2-2-(-3)) \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$

(d) $\vec{r} = 2\vec{b} - \frac{1}{2}\vec{c} = 2\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} (2\times 3 - 1/2\times(-2)) \\ (2\times 2 - 1/2\times(-3)) \end{bmatrix} = \begin{bmatrix} 7 \\ 11/2 \end{bmatrix}$

8.14. By inspection of sketch in Fig. 8.9 (left),

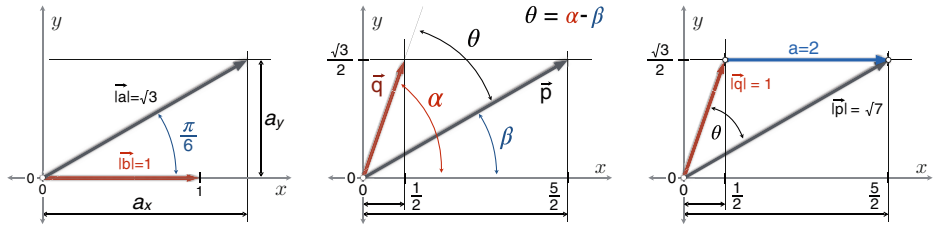


Fig. 8.9 Example P.8.14

$$\vec{a} = \vec{a_x} + \vec{a_y}$$
$$\vec{b} = \vec{b_x} + \vec{b_y}$$

where the horizontal and vertical projections are calculated as

$$|\vec{a_x}| = |\vec{a}| \cos \frac{\pi}{6} = \sqrt{3}\,\frac{\sqrt{3}}{2} = \frac{3}{2}$$
$$|\vec{a_y}| = |\vec{a}| \sin \frac{\pi}{6} = \sqrt{3}\,\frac{1}{2} = \frac{\sqrt{3}}{2}$$
$$|\vec{b_x}| = 1$$
$$|\vec{b_y}| = 0$$

Method 1: matrix forms of the two vectors are therefore

$$\vec{a} = \begin{bmatrix} 3/2 \\ \sqrt{3}/2 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then,

$$\vec{p} = \vec{a} + \vec{b} = \begin{bmatrix} 3/2 \\ \sqrt{3}/2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/2 \\ \sqrt{3}/2 \end{bmatrix}$$
$$\vec{q} = \vec{a} - \vec{b} = \begin{bmatrix} 3/2 \\ \sqrt{3}/2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

which leads to $\vec{p}$ and $\vec{q}$ as in Fig. 8.9 (center). By inspection of two right-angled triangles where $\vec{p}$ and $\vec{q}$ form their respective hypotenuses, angle θ between $\vec{p}$ and $\vec{q}$ is then

$$\theta = \alpha - \beta$$
$$\tan\alpha = \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \quad \therefore \quad \alpha = \arctan\sqrt{3} = \frac{\pi}{3} = 60^\circ$$
$$\tan\beta = \frac{\sqrt{3}/2}{5/2} = \frac{\sqrt{3}}{5} \quad \therefore \quad \alpha = \arctan\frac{\sqrt{3}}{5} = 19.1^\circ$$
$$\therefore$$
$$\theta = 40.89^\circ$$

Method 2: alternatively,

$$\vec{p} = \begin{bmatrix} 5/2 \\ \sqrt{3}/2 \end{bmatrix} \quad \therefore \quad |\vec{p}| = \sqrt{(5/2)^2 + (\sqrt{3}/2)^2} = \sqrt{7}$$
$$\vec{q} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \quad \therefore \quad |\vec{q}| = \sqrt{(1/2)^2 + (\sqrt{3}/2)^2} = 1$$

Then by the law of cosines for triangle formed by $\vec{p}$ and $\vec{q}$ (see Fig. 8.9 (right)), it follows that the third side of that triangle is $a = 2$. Now, by Pythagoras' theorem,

$$a^2 = b^2 + c^2 - 2bc\cos\theta \;\;\therefore\;\; 2^2 = (\sqrt{7})^2 + 1^2 - 2\sqrt{7}\,\cos\theta \;\;\therefore\;\; 2\sqrt{7}\theta = 4$$

$$\therefore\;\; \cos\theta = \frac{2}{\sqrt{7}} \;\;\therefore\;\; \theta = \arctan\frac{2}{\sqrt{7}} = 40.89^\circ$$

8.3 Linear Transformations

8.15. In its basic form, a *function* is equivalent to a machine that for any given number input x performs an operation $f(x)$ and produces the corresponding output y. Similarly, for any given vector input $\vec{v}$, *linear operation* $L(\vec{v})$ performs an operation and produces the corresponding vector output $\vec{w}$, as illustrated in Fig. 8.10 (left).

Another way to visualize linear transformation is to visualize the input vector being displaced in space until it overlaps with the output vector; see Fig. 8.10 (center). By doing so, however, linear operation $L(\vec{v})$ displaces *all* vectors in R^2 vector space, that is to say, it displaces *all the points on the surface*.

For example (see Fig. 8.10 (right)), while linear transformation $L(\vec{v})$ displaces basis vector $\vec{i} = (1, 0)$ into its transformed version $L(\vec{i}) = (1, 3)$, at the same time, basis vector $\vec{j} = (0, 1)$ is transformed into $L(\vec{j}) = (2, -1)$. Since all points on the surface are also simultaneously displaced, it may be imagined that the surface is "elastic" and that $L(\vec{v})$ "stretched" the whole surface. By that reasoning, now $L(\vec{i})$ and $L(\vec{j})$ represent basis vectors in the transformed space.

In summary, for a transformation to be *linear*, it must obey the following rules:

(a) After the transformation, all grid lines must stay straight.
(b) All the grid lines must stay parallel and equidistant because the distance between them must always be one basis vector.
(c) The origin point must not move.

As the consequence of the linearity, the *numerical form* of an arbitrary vector $\vec{x} = f(\vec{i}, \vec{j}\,)$ *does not change* in the transformed space, i.e., $L(\vec{x}) = f\big(L(\vec{i}\,), L(\vec{j}\,)\big)$.

8.16. By means of $L(\vec{x})$ transformation, $R2$ space as defined by $(\vec{i}, \vec{j}\,)$ (Fig. 8.11 (left)) may be transformed, for example, into $R2$ space as in Fig. 8.11 (right). The question is then: what is the form of linear transformation matrix L that causes that transformation?

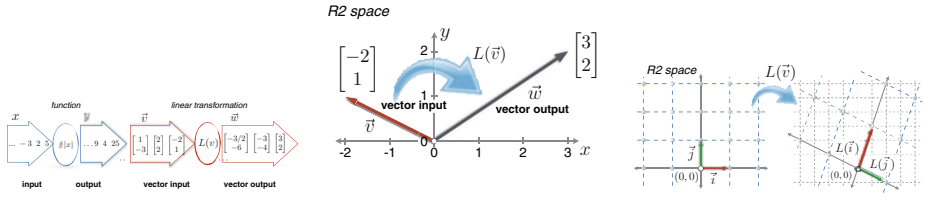


Fig. 8.10 Example P.8.15

Observe the relationship between the original and transformed greed lines. In order to better visualize it, the original grid is kept in the background of the transformed space. By definition, each square of the original grid represents unit size as defined by basis vectors $(\vec{i}, \vec{j})$. Next, by following the basis vectors $(\vec{i}, \vec{j})$, it can be deduced how the original R^2 space is rotated, flipped, and stretched so that basis vectors $(\vec{i}, \vec{j})$ are transformed into their new forms $L(\vec{i})$, $L(\vec{j})$.

By inspection of the transformed space grid, the expression for transformed $L(\vec{v})$ may be compared with the original vector $\vec{v}$. In order to be truly linear transformation, the numerical forms of these two equations must stay the same, i.e.,

$$\text{if in the original space,} \quad \vec{v} = -2\vec{i} + \vec{j}$$

$$\text{then in the transformed space,} \quad L(\vec{v}) = -2L(\vec{i}) + L(\vec{j})$$

At the same time, coordinates of basis vectors $L(\vec{i})$ and $L(\vec{j})$ in the transformed 2D space, *if expressed in the units of transformed space,* are

$$L(\vec{i}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad L(\vec{j}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

that is to say, by definition, they are orthonormal. However, *if measured in the units of original non-transformed space* where $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$, i.e., by using the background grid in Fig. 8.11 (right) to read the coordinates of $L(\vec{i})$, $L(\vec{j})$, and $L(\vec{v})$, it follows that

$$L(\vec{i}) = (1, 3), \quad L(\vec{j}) = (2, -1), \quad \text{and} \quad L(\vec{v}) = (0, -7)$$

Therefore, the transformation matrix L in this particular example is

$$L(\vec{i}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad L(\vec{j}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \therefore \quad L = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

and it can be verified the numerical form of $\vec{v}$ did not change, because

$$L(\vec{v}) = -2L(\vec{i}) + L(\vec{j}) = -2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \end{bmatrix} \quad \checkmark$$

Thus, in compact matrix form, transformation of an arbitrary $\vec{x}$ is

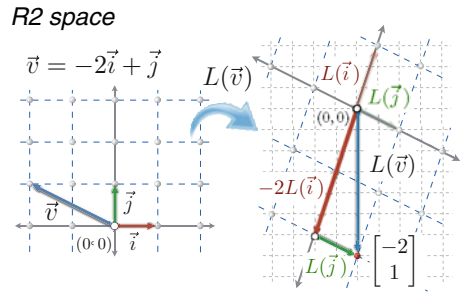


Fig. 8.11 Example P.8.16

$$\vec{x} = \begin{bmatrix} x_x \\ x_y \end{bmatrix} \quad \therefore \quad L(\vec{x}) = L\,\vec{x} = \left[L(\vec{i}\,)\; L(\vec{j}\,) \right] \vec{x} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_x \\ x_y \end{bmatrix}$$

In this interpretation, space transformations may be formalized with the help of the matrix–vector product.

8.17. Given data, and following the discussion in A.8.16, it follows that

$$L(\vec{i}\,) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad L(\vec{j}\,) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = (-1)\,\vec{i} + 2\vec{j}$$

The linear transformation form is unchanged as

$$L(\vec{x}) = L\,\vec{x} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = (-1)L(\vec{i}\,) + 2L(\vec{j}\,)$$

or, if measured in the units of original space:

$$= (-1) \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} (-1) \times 3 + 2 \times 1 \\ (-1) \times 1 + 2 \times 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

The relation between the original and transformed space is illustrated in Fig. 8.12.

8.18. Movement of basis vectors $(\vec{i}, \vec{j})$ is followed on graphs (see A.8.16) as:

(a) Rotation 90° counterclockwise: see Fig. 8.13 (left). Transformed basis vector coordinates as measured in non-transformed units are

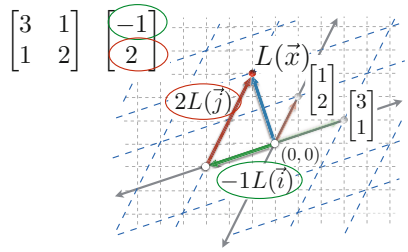
$$L(\vec{i}\,) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad L(\vec{j}\,) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \therefore \quad L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which is a transformation matrix that performs 90° rotation counterclockwise of *all* points in $R2$ space, therefore including basis vectors $\vec{i}$ and $\vec{j}$.

(b) Shear 45° clockwise: this operation of shear bends the space so that horizontal vectors are unchanged, while vertical vectors lean "−45°" (see Fig. 8.13 (center)); thus,

$$L(\vec{i}\,) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad L(\vec{j}\,) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \therefore \quad L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Fig. 8.12 Example P.8.17



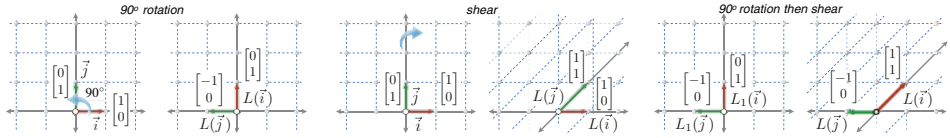


Fig. 8.13 Example P.8.18

which is a transformation matrix that performs shear of *all* points in R^2 space.

(c) First rotation 90° counterclockwise and then shear 45° clockwise: linear transformations may be performed one after another, where the order of transformations is *not* interchangeable. The first step is rotation L_1 and then sheer L_2; see Fig. 8.13 (right). The final position is then

$$L_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \quad L_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \therefore \quad L = L_2 L_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

which is a matrix that performs this rotation plus shear of *all* points in $R2$ space. Multiple transformations are formalized as being equivalent to the product of subsequent transformation matrices where *transformation that is done first is written on the right side*. Note that in this interpretation, L is a matrix that does both transformations *simultaneously*.

8.19. Given vectors $\vec{a} = (2, 1)$, $\vec{b} = (1, 0)$, and $\vec{m} = (9, 1)$, this type of problems may be solved by either geometric or numerical methods.

Method 1: create a triangle defined by $\vec{a}$ and $\vec{m}$ vectors; see Fig. 8.14. By inspection of vector addition graph, it is evident that

$$\vec{m} = \vec{a} + 7\vec{b}$$

Method 2: the general form of linear combination is

$$\vec{m} = n\vec{a} + k\vec{b}$$

where n and k are coefficients to be calculated; therefore,

$$\begin{bmatrix} 9 \\ 1 \end{bmatrix} = n \begin{bmatrix} 2 \\ 1 \end{bmatrix} + k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which is the matrix form of the following system of equations:

$$9 = 2n + 1\,k \tag{8.1}$$

$$1 = 1\,n + 0\,k \tag{8.2}$$

This system of linear equations may be solved by the elimination of variables method as

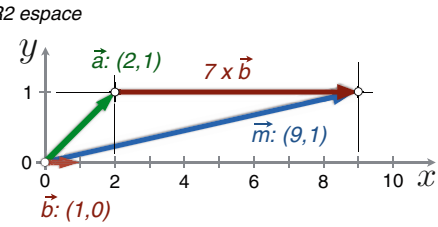


Fig. 8.14 Example P.8.19

$$\text{from (8.2)} : n = 1 \quad \therefore \quad \text{from (8.1)} : 9 = 2 + k \Rightarrow k = 7$$

In conclusion, $\vec{m} = \vec{a} + 7\vec{b}$.

8.20. Given vectors $\vec{u} = (3, -1)$, $\vec{v} = (1, -2)$, and $\vec{w} = (-1, 7)$, then, similar to A.8.19,

$$\vec{m} = \vec{u} + \vec{v} + \vec{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\vec{m} = n\vec{u} + k\vec{v}$$

$$\therefore$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = n \begin{bmatrix} 3 \\ -1 \end{bmatrix} + k \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

which is the matrix form of the following system of equations:

$$3 = 3n + 1k \tag{8.3}$$

$$4 = -1n - 2k \tag{8.4}$$

This system of linear equations may be solved by using multiple methods. For example,

Method 1: the elimination method

$$\text{from (8.3): } k = 3 - 3n \quad \therefore \quad \text{from (8.4): } 4 = -n - 2(3 - 3n) \Rightarrow n = 2$$
$$\text{from (8.4): } 4 = -2 - 2k \Rightarrow k = -3$$

Therefore, $\vec{m} = 2\vec{u} - 3\vec{v}$.

Method 2: Cramer's rule (see Sect. 8.5) gives

$$\left.\begin{aligned}\Delta &= \begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix} = -5 \neq 0 \\ \Delta_n &= \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = -10 \\ \Delta_k &= \begin{vmatrix} 3 & 3 \\ -1 & 4 \end{vmatrix} = 15 \end{aligned}\right\} \therefore \quad \begin{aligned} n &= \frac{\Delta_n}{\Delta} = \frac{-10}{-5} = 2 \\ k &= \frac{\Delta_k}{\Delta} = \frac{15}{-5} = -3 \end{aligned}$$

Therefore, $\vec{m} = 2\vec{u} - 3\vec{v}$.

8.4 Determinants

8.21. Given the second-order matrix A, its determinant is calculated as

$$|A| = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = (3 \times 2) - (0 \times 0) = 6 \neq 0$$

The geometrical interpretation is that this transformation A increases the surface of 2D space six times.

8.22. Given the second-order matrix A, its determinant is calculated as

$$|A| = \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = (1 \times (-1)) - (1 \times 2) = -3 \neq 0$$

The geometrical interpretation is that this transformation A increases the surface of 2D space three times and at the same time it is flipped over.

8.23. Given the second-order matrix A, its determinant is calculated as

$$|A| = \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = (4 \times 1) - (2 \times 2) = 0$$

Geometrically, this linear transformation A forces 2D space (i.e., a surface) to collapse onto 1D space (i.e., a line), i.e., one of the two coordinates is "flattened".

8.24. $\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 2 \times 0 - 1 \times (-1) = 1$

8.25. $\begin{vmatrix} \sqrt{3} & -3\sqrt{2} \\ \sqrt{3} & 2\sqrt{2} \end{vmatrix} = \sqrt{3} \times 2\sqrt{2} - \sqrt{3} \times (-3\sqrt{2}) = 2\sqrt{6} + 3\sqrt{6} = 5\sqrt{6}$

8.26. $\begin{vmatrix} 1 & i \\ -i & 1 \end{vmatrix} = 1 - (-i) \times i = 1 - 1 = 0$

8.27. $\begin{vmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{vmatrix} = \sin\alpha \sin\alpha - (-\cos\alpha)\cos\alpha = \sin^2\alpha + \cos^2\alpha = 1$

8.28. Given the third-order matrix A, its determinant may be calculated by

Method 1: cross-products as

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 4 \\ 5 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix} \begin{matrix} 2 & 3 \\ 5 & -2 \\ 1 & 2 \end{matrix}$$

$$\begin{aligned} &= +(2\times(-2)\times 3)+(3\times 1\times 1)+(4\times 5\times 2) \\ &\quad -(1\times(-2)\times 4)-(2\times 1\times 2)-(3\times 5\times 3) \\ &= -12+3+40+8-4-45 = \underline{-10} \end{aligned}$$

Method 2: expansion along the first row as

$$\Delta_3 = \begin{vmatrix} \textcircled{2} & 3 & 4 \\ 5 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & \textcircled{3} & 4 \\ 5 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 3 & \textcircled{4} \\ 5 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$\begin{aligned} &= (-1)^{1+1}\,2 \begin{vmatrix} -2 & 1 \\ 2 & 3 \end{vmatrix} + (-1)^{1+2}\,3 \begin{vmatrix} 5 & 1 \\ 1 & 3 \end{vmatrix} + (-1)^{1+3}\,4 \begin{vmatrix} 5 & -2 \\ 1 & 2 \end{vmatrix} \\ &= 2\times(-2\times 3-2\times 1)-3\times(5\times 3-1\times 1)+4\times(5\times 2-1\times(-2)) = \underline{-10} \end{aligned}$$

The geometrical interpretation is that this matrix transforms 3D space so that its volume is increased ten times and at the same time the space is "inverted inside-out" (as indicated by the negative sign of its determinant).

8.29. Given the third-order matrix A, its determinant may be calculated as

Method 1: cross-products as

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} \begin{matrix} 1 & 2 \\ 1 & 0 \\ 1 & 2 \end{matrix}$$

$$\begin{aligned} &= +(1\times 0\times 0)+(2\times 2\times 1)+(1\times 1\times 2) \\ &\quad -(1\times 0\times 1)-(2\times 2\times 1)-(0\times 1\times 2) \\ &= 6-4 = \underline{2} \end{aligned}$$

Method 2: expansion, for example, along the first column is

$$\Delta_3 = \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{1} \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ \cancel{1} & \cancel{0} & \cancel{2} \\ 1 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ \cancel{1} & \cancel{2} & \cancel{0} \end{vmatrix}$$
$$= (-1)^{1+1}\,1 \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} + (-1)^{2+1}\,1 \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + (-1)^{3+1}\,1 \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$$
$$= -4+2+4 = 2$$

8.30. Given the third-order matrix A, its determinant may be calculated as

Method 1: cross-products as

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 2 & 1 & 2 \\ 2 & 0 & 2 & 2 & 0 \\ 2 & 2 & 1 & 2 & 2 \end{vmatrix}$$
$$= +(1\times 0\times 1)+(2\times 2\times 2)+(2\times 2\times 2) \\ -(2\times 0\times 2)-(2\times 2\times 1)-(1\times 2\times 2)$$
$$= 16-8 = \underline{8}$$

Method 2: expansion, for example, along the second row is

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 2 \\ \cancel{2} & \cancel{0} & \cancel{2} \\ 2 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 2 \\ \cancel{2} & \cancel{0} & \cancel{2} \\ 2 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 2 \\ \cancel{2} & \cancel{0} & \cancel{2} \\ 2 & 2 & 1 \end{vmatrix}$$
$$= (-1)^{2+1}\,2 \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} + (-1)^{2+2}\,0 \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + (-1)^{2+3}\,2 \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$$
$$= 4+0+4 = \underline{8}$$

8.31. Given the third-order matrix A, its determinant may be calculated as

Method 1: cross-product

$$\Delta_3 = \begin{vmatrix} 3 & 2 & 1 & 3 & 2 \\ 4 & 5 & 6 & 4 & 5 \\ 8 & 9 & 7 & 8 & 9 \end{vmatrix}$$
$$= +(3\times 5\times 7)+(2\times 6\times 8)+(1\times 4\times 9) \\ -(8\times 5\times 1)-(9\times 6\times 3)-(7\times 4\times 2)$$
$$= (105+96+36)-(40+162+56)$$
$$= -21$$

Method 2: the cofactor expansion

$$\Delta_3 = \begin{vmatrix} \textcircled{3} & 2 & 1 \\ 4 & 5 & 6 \\ 8 & 9 & 7 \end{vmatrix} + \begin{vmatrix} 3 & \textcircled{2} & 1 \\ 4 & 5 & 6 \\ 8 & 9 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 2 & \textcircled{1} \\ 4 & 5 & 6 \\ 8 & 9 & 7 \end{vmatrix}$$
$$= (-1)^{1+1}\,3 \begin{vmatrix} 5 & 6 \\ 9 & 7 \end{vmatrix} + (-1)^{1+2}\,2 \begin{vmatrix} 4 & 6 \\ 8 & 7 \end{vmatrix} + (-1)^{1+3}\,1 \begin{vmatrix} 4 & 5 \\ 8 & 9 \end{vmatrix}$$
$$= 3 \times (5 \times 7 - 9 \times 6) - 2 \times (4 \times 7 - 8 \times 6) + 1 \times (4 \times 9 - 8 \times 5)$$
$$= -57 + 40 - 4 = -21$$

The volume of this 3D space is increased 21 times and "inverted inside-out".

8.32. Given the third-order matrix A, its determinant may be calculated as

Method 1: cross-product

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 1 & 2 & 3 \\ 3 & -1 & 2 & 3 & -1 \\ 1 & 1 & -3 & 1 & 1 \end{vmatrix}$$
$$= +(2 \times (-1) \times (-3)) + (3 \times 2 \times 1) + (1 \times 3 \times 1)$$
$$\quad -(1 \times (-1) \times 1) - (1 \times 2 \times 2) - ((-3) \times 3 \times 3)$$
$$= (6 + 6 + 3) - (-1 + 4 - 27)$$
$$= 39$$

Method 2: the cofactor expansion

$$\Delta_3 = \begin{vmatrix} \textcircled{2} & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 1 & -3 \end{vmatrix} + \begin{vmatrix} 2 & \textcircled{3} & 1 \\ 3 & -1 & 2 \\ 1 & 1 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 3 & \textcircled{1} \\ 3 & -1 & 2 \\ 1 & 1 & -3 \end{vmatrix}$$
$$= (-1)^{1+1}\,2 \begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} + (-1)^{1+2}\,3 \begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} + (-1)^{1+3}\,1 \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$$
$$= 2 \times \big((-1) \times (-3) - 1 \times 2\big) - 3 \times \big(3 \times (-3) - 1 \times 2\big) + 1 \times \big(3 \times 1 - 1 \times (-1)\big)$$
$$= 2 + 33 + 4 = 39$$

The geometrical interpretation is that this matrix transforms 3D space so that its volume is increased 39 times.

8.33. Given the third-order matrix A, its determinant may be calculated as

Method 1: cross-product

$$\Delta_3 = \begin{vmatrix} 5 & 0 & 4 & 5 & 0 \\ 8 & 0 & -7 & 8 & 0 \\ 3 & 2 & 1 & 3 & 2 \end{vmatrix}$$
$$= +(5 \times 0 \times 1) + (0 \times (-7) \times 3) + (4 \times 8 \times 2)$$
$$\quad -(3 \times 0 \times 4) - (2 \times (-7) \times 5) - (1 \times 8 \times 0)$$
$$= (0 + 0 + 64) - (0 - 70 - 0)$$
$$= 134$$

Method 2: any column or row could be chosen for cofactor expansion. It is beneficial, however, to choose one that contains zeros, so that the amount of work is reduced. In this example, the second column has two zeros, so the expansion is minimized as

$$\begin{aligned}
\Delta_3 &= \begin{vmatrix} 5 & 0 & 4 \\ 8 & 0 & -7 \\ 3 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 5 & 0 & 4 \\ 8 & 0 & -7 \\ 3 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 5 & 0 & 4 \\ 8 & 0 & -7 \\ 3 & 2 & 1 \end{vmatrix} \\
&= (-1)^{1+2}\, 0 \begin{vmatrix} 8 & -7 \\ 3 & 1 \end{vmatrix} + (-1)^{2+2}\, 0 \begin{vmatrix} 5 & 4 \\ 3 & 1 \end{vmatrix} + (-1)^{3+2}\, 2 \begin{vmatrix} 5 & 4 \\ 8 & -7 \end{vmatrix} \\
&= -0 + 0 - 2 \times \left(5 \times (-7) - 8 \times 4\right) \\
&= 134
\end{aligned}$$

The geometrical interpretation is that this matrix transforms 3D space so that its volume is increased 134 times.

8.34. Given the third-order matrix A, its determinant may be calculated as

Method 1: cross-product

$$\begin{aligned}
\Delta_3 = \begin{vmatrix} 1 & 3 & 5 & 1 & 3 \\ 7 & 9 & 11 & 7 & 9 \\ 13 & 15 & 17 & 13 & 15 \end{vmatrix} &= +(1 \times 9 \times 17) + (3 \times 11 \times 13) + (5 \times 7 \times 15) \\
&\quad -(13 \times 9 \times 5) - (15 \times 11 \times 1) - (17 \times 7 \times 3) \\
&= (153 + 429 + 525) - (585 + 165 + 357) \\
&= 1\,107 - 1\,107 = 0
\end{aligned}$$

Method 2: the cofactor expansion

$$\begin{aligned}
\Delta_3 &= \begin{vmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{vmatrix} \\
&= (-1)^{1+1}\, 1 \begin{vmatrix} 9 & 11 \\ 15 & 17 \end{vmatrix} + (-1)^{1+2}\, 3 \begin{vmatrix} 7 & 11 \\ 13 & 17 \end{vmatrix} + (-1)^{1+3}\, 5 \begin{vmatrix} 7 & 9 \\ 13 & 15 \end{vmatrix} \\
&= 1 \times (9 \times 17 - 15 \times 11) - 3 \times (7 \times 17 - 13 \times 11) + 5 \times (7 \times 15 - 13 \times 9) \\
&= 1\,(-12) - 3\,(-24) + 5\,(-12) = 0
\end{aligned}$$

The geometrical interpretation is that this matrix transforms 3D space so that at least one or more dimensions are lost (as indicated by zero determinant). That is to say, 3D space collapsed into 2D, 1D, or even point space.

8.35. Transformation of basis vectors $(\vec{i}, \vec{j})$ shows how the whole space is transformed. Given transformation

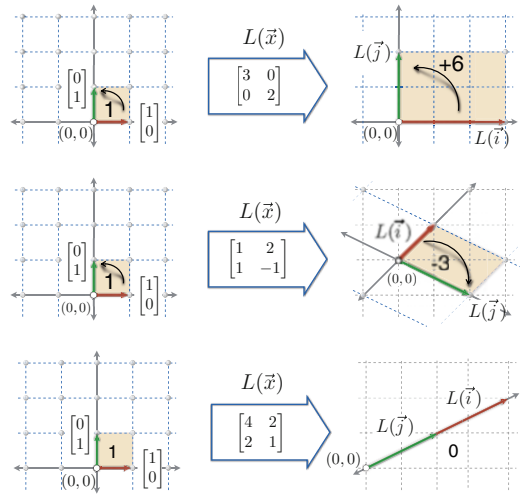


Fig. 8.15 Example P.8.35

$$L = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

note that this transformation simply multiplies the space by factor three in $\vec{i}$ direction (first column vector) and by factor two in $\vec{j}$ direction (second column vector) (see Fig. 8.15 (top)), because

$$L = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \therefore \quad L(\vec{i}) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad L(\vec{j}) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

It should be observed that:

(1) Relative position between basis vectors did not change, i.e., $\vec{j}$ is still on the left relative to $\vec{i}$. In other words, this transformation did not "flip" the space like a page in a book.
(2) The unit square area is increased by $3 \times 2 = 6$ positive factor. This (positive, zero, or negative) area multiplication factor is a geometrical interpretation of the determinant.

Therefore, this matrix's determinant is

$$|L| = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 3 \times 2 - 0 \times 0 = 6$$

8.36. Given transformation

$$L = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

performs multiple operations simultaneously. By following basis vectors, it could be deduced that 2D surface is rotated and flipped over; see Fig. 8.15 (center).

$$L = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \;\therefore\; L(\vec{i}\,) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad L(\vec{j}\,) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

and its determinant is

$$|L| = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = 1 \times (-1) - 1 \times 2 = -3$$

The geometrical interpretation is that the unity area is increased by a factor of "−3", that is to say, 2D surface is also "flipped over", which is understood by the negative determinant.

8.37. Given transformation

$$L(\vec{x}) = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \;\therefore\; L(\vec{i}\,) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{and} \quad L(\vec{j}\,) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \;\therefore\; L(\vec{i}\,) = 2\,L(\vec{j}\,)$$

and its determinant is

$$|L| = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = 4 \times 1 - 2 \times 2 = 0$$

Note that $L(\vec{i}\,)$ and $L(\vec{j}\,)$ are *collinear* because one is a simple multiple of the other, which is easily shown by factoring "2" as

$$L(\vec{i}\,) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\,L(\vec{j}\,)$$

After the transformation L, 2D surface collapsed into 1D line; see Fig. 8.15 (bottom). In the paper page analogy, imagine that 2D surface is sidewise rotated so that only its side is visible, i.e., line. This reduction of space is a geometrical interpretation of the case when the determinant equals zero. An alternative interpretation is that at least one of the equations within the system of equations in L is not independent.

8.5 Cramer's Rule

Reminder: The extended matrix that corresponds to the given system of equations is transformed into an extended unit matrix, so that the solution vector $[a, b, c]$ is found by inspection, for example, as

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right] \quad \begin{array}{l} 1\,x + 0\,y + 0\,z = a \\ 0\,x + 1\,y + 0\,z = b \\ 0\,x + 0\,y + 1\,z = c \end{array} \quad \begin{array}{l} \therefore \\ \\ \\ \end{array} \quad \begin{array}{l} x = a \\ y = b \\ z = c \end{array}$$

(continued)

(continued)
Systematically, one by one, each constant on the diagonal is transformed into "1" and all others into "0". Note the order of transformations for each constant on the diagonal, the lower triangle below the diagonal, and the upper triangle above the diagonal.

8.38. Given system of linear equations

$$\begin{aligned} 4x_1 - 3x_2 &= 0 \\ 2x_1 + 3x_2 &= 18 \end{aligned}$$

there are multiple methods to calculate its set of solutions. Each method has its advantages and disadvantages depending on the system's size and its coefficients. For the moment, note the column vector on the right side of equations that is colored.

Method 1: Cramer's rule is a formula to solve a system of linear equations with as many equations as unknowns. Therefore, for the second-order system of equations, there are three determinants to calculate: one "principal" (Δ) and one "sub-determinant" for each of the two variables (Δ_{x_1} and Δ_{x_2}) created by replacing its respective column with the column vector on the right side of the equations (colored).

$$A = \left[\begin{array}{cc|c} 4 & -3 & 0 \\ 2 & 3 & 18 \end{array}\right] \quad \therefore \quad \Delta = \begin{vmatrix} 4 & -3 \\ 2 & 3 \end{vmatrix} = 4 \times 3 - 2 \times (-3) = 18 \neq 0$$

(non-zero determinant implies the existence of a unique set of solutions)

$$\Delta_{x_1} = \begin{vmatrix} 0 & -3 \\ 18 & 3 \end{vmatrix} = 0 \times 3 - 18 \times (-3) = 3 \times 18$$

$$\Delta_{x_2} = \begin{vmatrix} 4 & 0 \\ 2 & 18 \end{vmatrix} = 4 \times 18 - 2 \times (0) = 4 \times 18$$

Then, by Cramer's rule,

$$x_1 = \frac{\Delta_{x_1}}{\Delta} = \frac{3 \times \cancel{18}}{\cancel{18}} = 3; \quad x_2 = \frac{\Delta_{x_1}}{\Delta} = \frac{4 \times \cancel{18}}{\cancel{18}} = 4$$

Method 2: the extended matrix rows' transformations (i.e., the right-side column vector is included). Note that rows are numbered as "(1)" and "(2)", while the associated arithmetic operations are simply "$+, -, \div, \times$". For example, "$\leftarrow (1) \div 4$" reads as "divide each coefficient in the first row on the left by four, including the colored vector column", or "$\leftarrow (2) - 2 \times (1)$" reads as "multiply each coefficient of the first row on the left by negative two and add to the second row on the left". The objective is to transform the systems' matrix into the unit matrix (i.e., where all diagonal coefficients equal one and all other coefficients equal zero). It is done by systematically converting each coefficient term, starting with the $(a_{1,1})$ coefficient, then the lower triangle coefficients, and finally the upper triangle coefficients.

$$A = \left[\begin{array}{cc|c} 4 & -3 & 0 \\ 2 & 3 & 18 \end{array}\right] \leftarrow (1) \div 4 = \left[\begin{array}{cc|c} 1 & -3/4 & 0 \\ 2 & 3 & 18 \end{array}\right] \leftarrow (2) - 2 \times (1)$$

$$= \left[\begin{array}{cc|c} 1 & -3/4 & 0 \\ 0 & 9/2 & 18 \end{array}\right] \leftarrow (2) \div 9/2 = \left[\begin{array}{cc|c} 1 & -3/4 & 0 \\ 0 & 1 & 4 \end{array}\right] \leftarrow (1) + 3/4 \times (2)$$

$$= \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

The last extended matrix form is an efficient way to write the following trivial system of equations:

$$1 \times x_1 - 0 \times x_2 = 3 \quad \Rightarrow x_1 = 3$$
$$0 \times x_1 + 1 \times x_2 = 4 \quad \Rightarrow x_2 = 4$$

Method 3: the variable substitution; it is an efficient hand method for small-size systems

$$4x_1 - 3x_2 = 0 \tag{8.5}$$
$$2x_1 + 3x_2 = 18 \tag{8.6}$$
$$\text{from (8.5)} \quad x_1 = \frac{3}{4}x_2 \tag{8.7}$$

then by substitution of x_1 in (8.6): $2\frac{3}{4}x_2 + 3x_2 = 18 \quad \therefore \quad x_2 = \dfrac{2 \times 18}{9} = 4$

and, by substitution of x_2 in (8.7): $x_1 = \dfrac{3}{4} \cdot 4 = 3$

8.39. Given second-order system,

$$x + 2y = -5$$
$$3x - y = 13$$

Method 1: Cramer's rule; the system of equations is written as a vector–matrix product, i.e.,

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 13 \end{bmatrix} \quad \therefore \quad \left.\begin{array}{l} \Delta = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -7 \\ \Delta_x = \begin{vmatrix} -5 & 2 \\ 13 & -1 \end{vmatrix} = -21 \\ \Delta_y = \begin{vmatrix} 1 & -5 \\ 3 & 13 \end{vmatrix} = 28 \end{array}\right\} \begin{array}{l} x = \dfrac{\Delta_x}{\Delta} = \dfrac{-21}{-7} = 3 \\ y = \dfrac{\Delta_y}{\Delta} = \dfrac{28}{-7} = -4 \end{array}$$

Method 2: or, for example, by the variable elimination,

$$x + 2y = -5$$
$$3x - y = 13 \quad \times 2$$

$$x + 2y = -5 \tag{8.8}$$
$$6x - 2y = 26 \tag{8.9}$$
$$\therefore \quad ((8.8) + (8.9))$$

$$x + 2y = -5$$
$$7x = 21 \quad \therefore \quad \underline{x = 3} \quad \therefore \quad \text{(back to (8.8))}$$

$$3 + 2y = -5 \quad \therefore \quad \underline{y = -4}$$

8.40. Given second-order system,

$$x + 2y = -8$$
$$2x - y = -1$$

Method 1: Cramer's rule; the system of equations is written as a vector–matrix product, i.e.,

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ -1 \end{bmatrix} \quad \therefore \quad \left.\begin{aligned} \Delta &= \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5 \\ \Delta_x &= \begin{vmatrix} -8 & 2 \\ -1 & -1 \end{vmatrix} = 10 \\ \Delta_y &= \begin{vmatrix} 1 & -8 \\ 2 & -1 \end{vmatrix} = 15 \end{aligned}\right\} \begin{aligned} x &= \frac{\Delta_x}{\Delta} = \frac{10}{-5} = -2 \\ y &= \frac{\Delta_y}{\Delta} = \frac{15}{-5} = -3 \end{aligned}$$

Method 2: or, for example, by the elimination method,

$$x + 2y = -8$$
$$2x - y = -1 \quad \times 2$$

$$x + 2y = -8$$
$$4x - 2y = -2 \quad \therefore$$

$$5x = -10 \quad \therefore \quad \underline{x = -2}$$
$$-2 + 2y = -8 \quad \therefore \quad \underline{y = -3}$$

Method 3: or, by the extended matrix transformations,

$$A = \left[\begin{array}{cc|c} 1 & 2 & -8 \\ 2 & -1 & -1 \end{array}\right] \begin{array}{l} \leftarrow (1) \ \checkmark \\ \leftarrow (2) - 2 \times (1) \end{array} = \left[\begin{array}{cc|c} 1 & 2 & -8 \\ 0 & -5 & 15 \end{array}\right] \leftarrow (2) \div (-5)$$

$$= \left[\begin{array}{cc|c} 1 & 2 & -8 \\ 0 & 1 & -3 \end{array}\right] \leftarrow (1) - 2 \times (2) = \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & -3 \end{array}\right]$$

The last extended matrix form is an efficient way to write the following trivial system of equations:

$$1 \times x - 0 \times y = -2 \Rightarrow x = -2$$
$$0 \times x + 1 \times y = -3 \Rightarrow y = -3$$

8.41. Given second-order system,

$$a + b = 7$$
$$2a + 2b = 14$$

Method 1: Cramer's rule; the system of equations is written as a vector–matrix product, i.e.,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix} \quad \therefore \quad \Delta = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$$

Therefore, the conclusion is that this system of equations does *not* have a unique set of solutions. It should be noted that the two equations are not independent; the second equation is simply two times the first equation, by consequence not independent.

8.42. Given third-order system,

$$\begin{array}{rrrl} (x) & (y) & (z) & \\ 5x - & 5y - & 15z = 40 & (1) \\ 4x - & 2y - & 6z = 19 & (2) \\ 3x - & 6y - & 17z = 41 & (3) \end{array}$$

Method 1: the extended matrix transformations,

$$\begin{array}{r} 5x - 5y - 15z = 40 \\ 4x - 2y - 6z = 19 \\ 3x - 6y - 17z = 41 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 5 & -5 & -15 & 40 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{array}\right]$$

Therefore,

$$\left[\begin{array}{ccc|c} 5 & -5 & -15 & 40 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{array}\right] \leftarrow (1) \div [5] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -3 & 8 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{array}\right] \begin{array}{l} \leftarrow (2) - 4 \times (1) \\ \leftarrow (3) - 3 \times (1) \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 8 \\ 0 & 2 & 6 & -13 \\ 0 & -3 & -8 & 17 \end{array}\right] \leftarrow (2) \div [2] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -3 & 8 \\ 0 & 1 & 3 & -13/2 \\ 0 & -3 & -8 & 17 \end{array}\right] \leftarrow (3) + 3 \times (2) \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 8 \\ 0 & 1 & 3 & -13/2 \\ 0 & 0 & 1 & -5/2 \end{array}\right] \leftarrow (1) + (2) \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 3 & -13/2 \\ 0 & 0 & 1 & -5/2 \end{array}\right] \leftarrow (2) - 3 \times (3) \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -5/2 \end{array}\right] \quad \therefore \quad \begin{array}{l} 1x + 0y + 0z = 3/2 \\ 0x + 1y + 0z = 1 \\ 0x + 0y + 1z = -5/2 \end{array} \quad \therefore \quad \{x, y, z\} = \left\{\frac{3}{2}, 1, -\frac{5}{2}\right\}$$

Method 2: or, by Cramer's rule,

$$\Delta = \begin{vmatrix} 5 & -5 & -15 & 5 & -5 \\ 4 & -2 & -6 & 4 & -2 \\ 3 & -6 & -17 & 3 & -6 \end{vmatrix}$$

$$\begin{array}{ll} = +(5 \times (-2) \times (-17)) & -(3 \times (-2) \times (-15)) \\ +((-5) \times -(6) \times 3) & -((-6) \times (-6) \times 5) \\ +((-15) \times 4 \times (-6)) & -((-17) \times 4 \times (-5)) \end{array}$$

$$= (620) - (610) = 10 \neq 0$$

$$\Delta_x = \begin{vmatrix} 40 & -5 & -15 & 40 & -5 \\ 19 & -2 & -6 & 19 & -2 \\ 41 & -6 & -17 & 41 & -6 \end{vmatrix}$$

$$\begin{array}{ll} = +(40 \times (-2) \times (-17)) & -(41 \times (-2) \times (-15)) \\ +((-5) \times (-6) \times 41) & -((-6) \times (-6) \times 40) \\ +((-15) \times 19 \times (-6)) & -((-17) \times 19 \times (-5)) \end{array}$$

$$= (4300) - (4285) = 15$$

$$\Delta_y = \begin{vmatrix} 5 & 40 & -15 & 5 & 40 \\ 4 & 19 & -6 & 4 & 19 \\ 3 & 41 & -17 & 3 & 41 \end{vmatrix}$$

$$\begin{array}{ll} = +(5 \times 19 \times (-17)) & -(3 \times 19 \times (-15)) \\ +(40 \times (-6) \times 3) & -(41 \times (-6) \times 5) \\ +((-15) \times 4 \times 41) & -((-17) \times 4 \times 40)) \end{array}$$

$$= (-4795) - (-4805) = 10$$

$$\Delta_z = \begin{vmatrix} 5 & -5 & 40 & 5 & -5 \\ 4 & -2 & 19 & 4 & -2 \\ 3 & -6 & 41 & 3 & -6 \end{vmatrix}$$

$$\begin{array}{ll} = +(5 \times (-2) \times 41) & -(3 \times (-2) \times 40) \\ +((-5) \times 19 \times 3) & -((-6) \times 19 \times 5) \\ +(40 \times 4 \times (-6)) & -(41 \times 4 \times (-5))) \end{array}$$

$$= (-1655) - (-1630) = -25$$

Therefore,

$$x = \frac{\Delta_x}{\Delta} = \frac{15}{10} = \frac{3}{2}; \quad y = \frac{\Delta_y}{\Delta} = \frac{10}{10} = 1; \quad z = \frac{\Delta_z}{\Delta} = \frac{-25}{10} = -\frac{5}{2}$$

Method 3: the cofactor expansion to calculate the determinants and then apply Cramer's rule. Note that numbers in the third column are rather large, so

$$\begin{aligned}\Delta &= \begin{vmatrix} 5 & -5 & -15 \\ 4 & -2 & -6 \\ 3 & -6 & -17 \end{vmatrix} + \begin{vmatrix} 5 & -5 & -15 \\ 4 & -2 & -6 \\ 3 & -6 & -17 \end{vmatrix} + \begin{vmatrix} 5 & -5 & -15 \\ 4 & -2 & -6 \\ 3 & -6 & -17 \end{vmatrix} \\ &= (-1)^{1+3}(-15)\begin{vmatrix} 4 & -2 \\ 3 & -6 \end{vmatrix} + (-1)^{2+3}(-6)\begin{vmatrix} 5 & -5 \\ 3 & -6 \end{vmatrix} + (-1)^{3+3}(-17)\begin{vmatrix} 5 & -5 \\ 4 & -2 \end{vmatrix} \\ &= -15\times(-18)+6\times(-15)-17\times 10 = 10\end{aligned}$$

$$\begin{aligned}\Delta_x &= \begin{vmatrix} 40 & -5 & -15 \\ 19 & -2 & -6 \\ 41 & -6 & -17 \end{vmatrix} + \begin{vmatrix} 40 & -5 & -15 \\ 19 & -2 & -6 \\ 41 & -6 & -17 \end{vmatrix} + \begin{vmatrix} 40 & -5 & -15 \\ 19 & -2 & -6 \\ 41 & -6 & -17 \end{vmatrix} \\ &= (-1)^{1+3}(-15)\begin{vmatrix} 19 & -2 \\ 41 & -6 \end{vmatrix} + (-1)^{2+3}(-6)\begin{vmatrix} 40 & -5 \\ 41 & -6 \end{vmatrix} + (-1)^{3+3}(-17)\begin{vmatrix} 40 & -5 \\ 19 & -2 \end{vmatrix} \\ &= -15\times(-32)+6\times(-35)-17\times 15 = 15\end{aligned}$$

$$\begin{aligned}\Delta_y &= \begin{vmatrix} 5 & 40 & -15 \\ 4 & 19 & -6 \\ 3 & 41 & -17 \end{vmatrix} + \begin{vmatrix} 5 & 40 & -15 \\ 4 & 19 & -6 \\ 3 & 41 & -17 \end{vmatrix} + \begin{vmatrix} 5 & 40 & -15 \\ 4 & 19 & -6 \\ 3 & 41 & -17 \end{vmatrix} \\ &= (-1)^{1+3}(-15)\begin{vmatrix} 4 & 19 \\ 3 & 41 \end{vmatrix} + (-1)^{2+3}(-6)\begin{vmatrix} 5 & 40 \\ 3 & 41 \end{vmatrix} + (-1)^{3+3}(-17)\begin{vmatrix} 5 & 40 \\ 4 & 19 \end{vmatrix} \\ &= -15\times 107+6\times 85-17\times(-65) = 10\end{aligned}$$

$$\begin{aligned}\Delta_z &= \begin{vmatrix} 5 & -5 & 40 \\ 4 & -2 & 19 \\ 3 & -6 & 41 \end{vmatrix} + \begin{vmatrix} 5 & -5 & 40 \\ 4 & -2 & 19 \\ 3 & -6 & 41 \end{vmatrix} + \begin{vmatrix} 5 & -5 & 40 \\ 4 & -2 & 19 \\ 3 & -6 & 41 \end{vmatrix} \\ &= (-1)^{1+3}\,40\begin{vmatrix} 4 & -2 \\ 3 & -6 \end{vmatrix} + (-1)^{2+3}\,19\begin{vmatrix} 5 & -5 \\ 3 & -6 \end{vmatrix} + (-1)^{3+3}\,41\begin{vmatrix} 5 & -5 \\ 4 & -2 \end{vmatrix} \\ &= 40\times(-18)-19\times(-15)+41\times(10) = -25\end{aligned}$$

Again,

$$x = \frac{\Delta_x}{\Delta} = \frac{15}{10} = \frac{3}{2}; \quad y = \frac{\Delta_y}{\Delta} = \frac{10}{10} = 1; \quad z = \frac{\Delta_z}{\Delta} = \frac{-25}{10} = -\frac{5}{2}$$

Often, it is a simply a matter of preference which method is used.

8.43. Given third-order system of equations,

$$\begin{aligned} 2x - 3y - z &= 5 \\ x + y + 2z &= 7 \\ 2x - y - z &= 1 \end{aligned}$$

Method 1: for example, by Cramer's rule,

$$\Delta = \begin{vmatrix} 2 & -3 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & -1 \end{vmatrix} \begin{matrix} 2 & -3 \\ 1 & 1 \\ 2 & -1 \end{matrix} \begin{aligned} &= +(2\times 1\times(-1)) + (-3\times 2\times 2) + ((-1)\times 1\times(-1)) \\ &\quad -(2\times 1\times(-1)) - ((-1)\times 2\times 2) - ((-1)\times 1\times(-3)) \\ &= (-13) - (-3) = -10 \neq 0 \end{aligned}$$

$$\Delta_x = \begin{vmatrix} 5 & -3 & -1 \\ 7 & 1 & 2 \\ 1 & -1 & -1 \end{vmatrix} \begin{matrix} 5 & -3 \\ 7 & 1 \\ 1 & -1 \end{matrix} \begin{aligned} &= +(5\times 1\times(-1)) + ((-3)\times 2\times 1) + ((-1)\times 7\times(-1)) \\ &\quad -(1\times 1\times(-1)) - ((-1)\times 2\times 5) - ((-1)\times 7\times(-3)) \\ &= (-4) - (10) = -14 \end{aligned}$$

$$\Delta_y = \begin{vmatrix} 2 & 5 & -1 \\ 1 & 7 & 2 \\ 2 & 1 & -1 \end{vmatrix} \begin{matrix} 2 & 5 \\ 1 & 7 \\ 2 & 1 \end{matrix} \begin{aligned} &= +(2\times 7\times(-1)) + (5\times 2\times 2) + ((-1)\times 1\times 1) \\ &\quad -(2\times 7\times(-1)) - (1\times 2\times 2) - ((-1)\times 1\times 5) \\ &= (5) - (-15) = 20 \end{aligned}$$

$$\Delta_z = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 1 & 7 \\ 2 & -1 & 1 \end{vmatrix} \begin{matrix} 2 & -3 \\ 1 & 1 \\ 2 & -1 \end{matrix} \begin{aligned} &= +(2\times 1\times 1) + ((-3)\times 7\times 2) + (5\times 1\times(-1)) \\ &\quad -(2\times 1\times 5) - ((-1)\times 7\times 2) - (1\times 1\times(-3)) \\ &= (-45) - (-7) = -38 \end{aligned}$$

Then,

$$x = \frac{\Delta_x}{\Delta} = \frac{-14}{-10}\frac{7}{5} = \frac{7}{5}; \qquad y = \frac{\Delta_y}{\Delta} = \frac{20}{-10}\frac{-2}{} = -2; \qquad z = \frac{\Delta_z}{\Delta} = \frac{-38}{-10}\frac{19}{5} = \frac{19}{5}$$

8.44. Given third-order system of equations,

$$\begin{aligned} x + y + z &= 3 \\ x - y + z &= 5 \end{aligned}$$

$$-x + y - z = 10$$

Method 1: by Cramer's rule,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix} \begin{matrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{matrix} = 1 - 1 + 1 - 1 - 1 + 1 = 0$$

which is to say that this system does not have a unique set of solutions. It should be noted that the second and third equations' coefficients are not independent; there is a negative one factor between the two.

Method 2: it may be more obvious that the second and third equations are contradictory if the system is solved by substitution as

$$\begin{aligned} x + y + z = 3 &\Rightarrow x = 3 - y - z \\ x - y + z = 5 & \\ -x + y - z = 10 & \end{aligned}$$

$$\begin{aligned} (3 - y - z) - y + z = 5 &\Rightarrow -2y + 3 = 5 \quad \therefore \quad \underline{y = -1} \\ -(3 - y - z) + y - z = 10 &\Rightarrow 2y - 3 = 10 \end{aligned}$$

$$2(-1) - 3 = 10 \Rightarrow \underline{-5 = 10} \quad \times$$

which, obviously, is false. Therefore, there is no unique set of solutions to this system.

8.45. Given third-order system of equations, note the positions of "0" coefficients that keep places of the "missing" variables

$$\begin{aligned} x + y + z &= 36 \\ 2x + 0y - z &= -17 \\ 6x + 0y - 5z &= 7 \end{aligned}$$

Method 1: for example, by the matrix transformations,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 2 & 0 & -1 & -17 \\ 6 & 0 & -5 & 7 \end{array}\right] \begin{matrix} \leftarrow (1) \ \checkmark \\ \leftarrow (2) - 2 \times (1) \\ \\ \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & -2 & -3 & -89 \\ 6 & 0 & -5 & 7 \end{array}\right] \begin{matrix} \\ \leftarrow (2) \div (-2) \\ \leftarrow (3) - 6 \times (1) \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & 1 & 3/2 & 89/2 \\ 0 & -6 & -11 & -209 \end{array}\right] \leftarrow (3) + 6 \times (2) \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & 1 & 3/2 & 89/2 \\ 0 & 0 & -2 & 58 \end{array}\right] \leftarrow (3) \div [-2]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & 1 & 3/2 & 89/2 \\ 0 & 0 & 1 & -29 \end{array}\right] \leftarrow (1)-(2) \qquad \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & -17/2 \\ 0 & 1 & 3/2 & 89/2 \\ 0 & 0 & 1 & -29 \end{array}\right] \begin{array}{l} \leftarrow (1)+1/2(3) \\ \leftarrow (2)-3/2(3) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -23 \\ 0 & 1 & 0 & 88 \\ 0 & 0 & 1 & -29 \end{array}\right] \quad \therefore \qquad \{x, y, z\} = \{-23, 88, -29\}$$

8.46. Given third-order system of equations,

$$\begin{aligned} x + y &= 7 \\ y + z &= 8 \\ -x + 2z &= 7 \end{aligned}$$

Method 1: for example, by the matrix transformations,

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 1 & 8 \\ -1 & 0 & 2 & 7 \end{array}\right] \begin{array}{l} \leftarrow (1)\ \checkmark \\ \leftarrow (2)\ \checkmark \\ \leftarrow (3)+(1) \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 1 & 8 \\ 0 & 1 & 2 & 14 \end{array}\right] \leftarrow (3)-(2)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 6 \end{array}\right] \leftarrow (1)-(2) \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 6 \end{array}\right] \leftarrow (1)+(3)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 6 \end{array}\right] \leftarrow (2)-(3) \qquad \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$\therefore \qquad \{x, y, z\} = \{5, 2, 6\}$$

Method 2: by Cramer's rule,

$$\Delta = \left|\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & -1 & 0 \end{array}\right| = 2 + (-1) + 0 - 0 - 0 - 0 = 1 \neq 0$$

$$\Delta_x = \left|\begin{array}{ccc|cc} 7 & 1 & 0 & 7 & 1 \\ 8 & 1 & 1 & 8 & 1 \\ 7 & 0 & 2 & 7 & 0 \end{array}\right| = 14 + 7 + 0 - 0 - 0 - 16 = 5$$

$$\Delta_y = \left|\begin{array}{ccc|cc} 1 & 7 & 0 & 1 & 7 \\ 0 & 8 & 1 & 0 & 8 \\ -1 & 7 & 2 & -1 & 7 \end{array}\right| = 16 + (-7) + 0 - 0 - 7 - 0 = 2$$

$$\Delta_z = \left|\begin{array}{ccc|cc} 1 & 1 & 7 & 1 & 1 \\ 0 & 1 & 8 & 0 & 1 \\ -1 & 0 & 7 & -1 & 0 \end{array}\right| = 7 + (-8) + 0 - (-7) - 0 - 0 = 6$$

$$\therefore \quad x = \frac{\Delta_x}{\Delta} = \frac{5}{1} = 5 \quad y = \frac{\Delta_y}{\Delta} = \frac{2}{1} = 2 \quad z = \frac{\Delta_z}{\Delta} = \frac{6}{1} = 6$$

$$\text{that is to say, } \vec{v} = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix} \quad \text{or} \quad (x, y, z) = (5, 2, 6)$$

8.47. Given third-order system of equations,

$$\begin{aligned} 5x - 13y + 13z &= 8 \\ x - 3y + z &= 4 \\ -x + 2y - 5z &= 3 \end{aligned}$$

Method 1: by Cramer's rule,

$$\begin{aligned} \Delta &= \begin{vmatrix} 5 & -13 & 13 \\ 1 & -3 & 1 \\ -1 & 2 & -5 \end{vmatrix} \\ &= (-1)^{1+1}\, 5 \begin{vmatrix} -3 & 1 \\ 2 & -5 \end{vmatrix} + (-1)^{1+2}\, (-13) \begin{vmatrix} 1 & 1 \\ -1 & -5 \end{vmatrix} + (-1)^{1+3}\, 13 \begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} \\ &= 5 \times 13 + 13 \times (-4) + 13 \times (-1) = 0 \end{aligned}$$

which is to say that this system does not have a unique set of solutions.

8.48. Two points A, B must be found on the linear equation $f(x)$ simultaneously; there is one equation corresponding to each point (x, y); therefore,

$$\begin{aligned} & & \therefore \quad & ax + b = f(x) \\ \underline{(x, y) = (-2, 20)} & & \therefore \quad & a(-2) + b = 20 \\ \underline{(x, y) = (1, 5)} & & \therefore \quad & a(1) + b = 5 \end{aligned}$$

Therefore, the matrix form of the problem is

$$\begin{array}{rrr} a & b & f(x) \\ -2a & +b = & 20 \\ a & +b = & 5 \end{array}$$

Method 1: for example, by substitution,

$$\begin{aligned} -2a + b &= 20 \\ a + b &= 5 \quad \Rightarrow a = 5 - b \end{aligned}$$

$$-2(5-b)+b=20 \;\Rightarrow 3b-10=20 \;\therefore\; \underline{b=10} \;\therefore\; \underline{a=-5}$$
$$\therefore\; \{a,b\}=\{-5,10\}$$

which is to say that $f(x)=-5x+10$.

Verification:

$$f(-2)=-5(-2)+10=20 \;\checkmark$$
$$f(1)=-5(1)+10=5 \;\checkmark$$

Note that, given two points, there is only one unique linear function that crosses both points. However, there are infinitely many higher-order nonlinear functions that cross these two points. It may be shown, for example, by deriving quadratic fitting function as

$$\therefore\; ax^2+bx+c=f(x)$$
$$\underline{(x,y)=(-2,20)} \;\therefore\; a(-2)^2+b(-2)+c=20$$
$$\underline{(x,y)=(1,5)} \;\therefore\; a(1)^2+b(1)+c=5$$

Therefore, the matrix form of the problem is

$$\begin{array}{rrrr} a & b & c & f(x) \\ 4a & -2b & +c = & 20 \\ a & +b & +c = & 5 \end{array}$$

that is to say, there are there unknown variables and only two equations; consequently, the system does not have a unique set of solutions. That being the case, one variable is always dependent on the other two, as

$$4a-2b+c=20$$
$$a+b+c=5 \;\Rightarrow a=5-b-c$$

$$4(5-b-c)-2b+c=20 \;\Rightarrow \underline{b=-\frac{c}{2}}$$

$$a=5-\left(-\frac{c}{2}\right)-c \;\therefore\; \underline{a=5-\frac{c}{2}}$$
$$\therefore\; \{a,b,c\}=\left\{5-\frac{c}{2},-\frac{c}{2},c\right\}$$

which is to say, by consequence of c being a parameter, that there are infinitely many quadratic functions of the form $f(x)=ax^2+bx+c$ that cross the two given points. A couple of examples (see Fig. 8.16) are as follows:

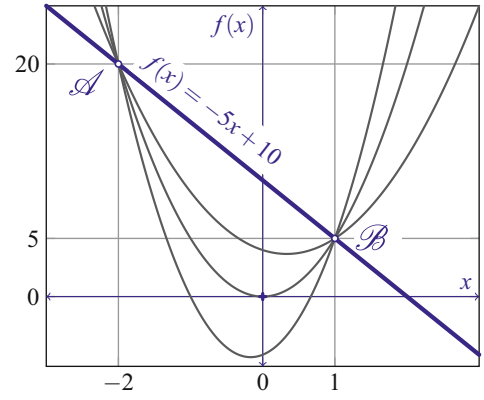


Fig. 8.16 Example P.8.48

$$\underline{c=-1}\ :\quad f_1(x)=ax^2+bx+c=\frac{11}{2}x^2+\frac{1}{2}x-1$$
$$\underline{c=0}\ :\quad f_2(x)=ax^2+bx+c=5x^2$$
$$\underline{c=4}\ :\quad f_3(x)=ax^2+bx+c=3x^2-2x+4$$
$$\text{etc.}$$

In conclusion, there are infinitely many curves that cross given two points, but linear function is the only unique fitting "curve".

8.49. All three points A, B, C must be found on the quadratic equation $f(x)$ simultaneously; there is one equation for each point (x, y); therefore,

$$\therefore\quad ax^2+bx+c=f(x)$$
$$\underline{(x,y)=(-2,20)}\quad\therefore\quad a(-2)^2+b(-2)+c=20$$
$$\underline{(x,y)=(1,5)}\quad\therefore\quad a(1)^2+b(1)+c=5$$
$$\underline{(x,y)=(3,25)}\quad\therefore\quad a(3)^2+b(3)+c=25$$

Therefore, the matrix form of the problem is

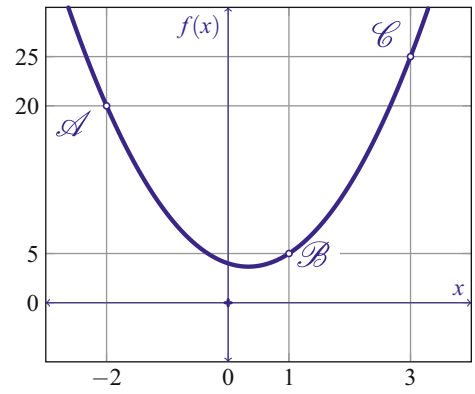
$$\begin{array}{rrrl} a & b & c & \\ 4a & -2b & +c & =20 \\ a & +b & +c & =5 \\ 9a & +3b & +c & =25 \end{array}$$

that is to say,

$$A\,\vec{v}=\vec{b}$$

where

Fig. 8.17 Example P.8.49



$$A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 20 \\ 5 \\ 25 \end{bmatrix}$$

Method 1: for example, by matrix transformations. Note that the order of equations (i.e., rows) may be changed (e.g., (1) ↔ (2)) because the order of equations in the original system is arbitrary,

$$\left[\begin{array}{ccc|c} 4 & -2 & 1 & 20 \\ 1 & 1 & 1 & 5 \\ 9 & 3 & 1 & 25 \end{array}\right] \quad (1) \leftrightarrow (2) \qquad \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 4 & -2 & 1 & 20 \\ 9 & 3 & 1 & 25 \end{array}\right] \quad \leftarrow (2) - 4 \times (1)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -6 & -3 & 0 \\ 9 & 3 & 1 & 25 \end{array}\right] \quad \leftarrow (2) \div (-6) \qquad \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 1/2 & 0 \\ 9 & 3 & 1 & 25 \end{array}\right] \quad \leftarrow (3) - 9 \times (1)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 1/2 & 0 \\ 0 & -6 & -8 & -20 \end{array}\right] \quad \leftarrow (3) + 6 \times (2) \qquad \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & -5 & -20 \end{array}\right] \quad \leftarrow (3) \div (-5)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \leftarrow (1) - (2) \qquad \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 5 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \leftarrow (2) - 1/2(3)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \leftarrow (1) - 1/2(3) \qquad \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \therefore \quad \{a, b, c\} = \{3, -2, 4\}$$

which is to say that $f(x) = 3x^2 - 2x + 4$; see Fig. 8.17. Following the discussion in A.8.48, there are infinitely many higher-order parametric curves that cross these three points; however, being parametrized does not represent a unique set of solutions.

8.50. To say that two or more vectors are linearly independent is to say that they are not parallel, or to say that their first derivatives are not the same (see chapters on calculus) , or to say that the system's

determinant is $\Delta \neq 0$. Given

$$\vec{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

(a) The linear combination is

$$\vec{c} = k\vec{a} + n\vec{b}$$

$$\therefore$$

$$\begin{bmatrix} 7 \\ -4 \end{bmatrix} = k \begin{bmatrix} 3 \\ -2 \end{bmatrix} + n \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

which is the matrix form of the following system of equations:

$$7 = 3k - 2n$$
$$-4 = -2k + n$$

Therefore, the determinant is calculated as

$$\Delta = \begin{vmatrix} 3 & -2 \\ -2 & 1 \end{vmatrix} = \big(3 \times 1\big) - \big(-2 \times (-2)\big) = -1 \neq 0$$

which is to say that equations (i.e., vectors) are indeed independent. More precisely, Cramer's rule gives

$$\left.\begin{aligned} \Delta_k &= \begin{vmatrix} 7 & -2 \\ -4 & 1 \end{vmatrix} = -1 \\ \Delta_n &= \begin{vmatrix} 3 & 7 \\ -2 & -4 \end{vmatrix} = 2 \end{aligned}\right\} \quad \therefore \quad \begin{aligned} k &= \frac{\Delta_k}{\Delta} = \frac{-1}{-1} = 1 \\ n &= \frac{\Delta_n}{\Delta} = \frac{2}{-1} = -2 \end{aligned}$$

which is to say, $\vec{c} = \vec{a} - 2\vec{b}$.

(b) Given

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

The linear combination is

$$\vec{c} = k\vec{a} + n\vec{b} \quad \therefore \quad \begin{bmatrix} 0 \\ -2 \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + n \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

which is the matrix form of the following system of equations:

$$0 = k + 3n$$
$$-2 = 2k + 4n$$

Therefore, the determinant is calculated as

$$\Delta = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2 \neq 0$$

which is to say that equations (i.e., vectors) are indeed independent. Cramer's rule gives

$$\left.\begin{aligned} \Delta_k &= \begin{vmatrix} 0 & 3 \\ -2 & 4 \end{vmatrix} = 6 \\ \Delta_n &= \begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix} = -2 \end{aligned}\right\} \quad \therefore \quad \begin{aligned} k &= \frac{\Delta_k}{\Delta} = \frac{6}{-2} = -3 \\ n &= \frac{\Delta_n}{\Delta} = \frac{-2}{-2} = 1 \end{aligned}$$

which is to say, $\vec{c} = -3\vec{a} + \vec{b}$.

8.51. To say that two or more vectors are linearly independent is to say that they are not parallel, or to say that their first derivatives are not the same, or to say that the system's determinant is $\Delta \neq 0$. Given

$$\vec{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

(a) The linear combination is

$$\vec{c} = k\vec{a} + n\vec{b}$$

$$\therefore$$

$$\begin{bmatrix} 7 \\ -4 \end{bmatrix} = k \begin{bmatrix} 3 \\ -2 \end{bmatrix} + n \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

which is the matrix form of the following system of equations:

$$\begin{aligned} 7 &= 3k - 2n \\ -4 &= -2k + n \end{aligned}$$

Therefore, the determinant is calculated as

$$\Delta = \begin{vmatrix} 3 & -2 \\ -2 & 1 \end{vmatrix} = \big(3 \times 1\big) - \big(-2 \times (-2)\big) = -1 \neq 0$$

which is to say that equations (i.e., vectors) are indeed independent. More precisely, Cramer's rule gives

$$\left.\begin{aligned} \Delta_k &= \begin{vmatrix} 7 & -2 \\ -4 & 1 \end{vmatrix} = -1 \\ \Delta_n &= \begin{vmatrix} 3 & 7 \\ -2 & -4 \end{vmatrix} = 2 \end{aligned}\right\} \quad \therefore \quad \begin{aligned} k &= \frac{\Delta_k}{\Delta} = \frac{-1}{-1} = 1 \\ n &= \frac{\Delta_n}{\Delta} = \frac{2}{-1} = -2 \end{aligned}$$

which is to say, $\vec{c} = \vec{a} - 2\vec{b}$.

(b) Given

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

The linear combination is

$$\vec{c} = k\vec{a} + n\vec{b} \quad \therefore \quad \begin{bmatrix} 0 \\ -2 \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + n \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

which is the matrix form of the following system of equations:

$$0 = k + 3n$$
$$-2 = 2k + 4n$$

Therefore, the determinant is calculated as

$$\Delta = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2 \neq 0$$

which is to say that equations (i.e., vectors) are indeed independent. Cramer's rule gives

$$\left.\begin{aligned} \Delta_k &= \begin{vmatrix} 0 & 3 \\ -2 & 4 \end{vmatrix} = 6 \\ \Delta_n &= \begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix} = -2 \end{aligned}\right\} \quad \therefore \quad \begin{aligned} k &= \frac{\Delta_k}{\Delta} = \frac{6}{-2} = -3 \\ n &= \frac{\Delta_n}{\Delta} = \frac{-2}{-2} = 1 \end{aligned}$$

which is to say, $\vec{c} = -3\vec{a} + \vec{b}$.

8.52. Parallel vectors are said to be "collinear" (i.e., angle between them is either zero or π), as oppose to "orthogonal" (i.e., angle between them is $\pm\pi/2$) or any other arbitrary angle between them. Recalling that the slope of a linear function is calculated as its first derivative, then another method to establish spatial relationship (i.e., parallel, orthogonal, etc.) between two vectors is to calculate and compare their respective first derivatives; see chapters on calculus.

Reminder: One or more vectors are linearly independent if they are *not collinear*. One possible way to formalize that statement is to write

$$\text{if} \quad \vec{x} = k\vec{y} \quad \Rightarrow \quad (\vec{x}, \vec{y}) \quad \text{are collinear}$$

where k is the multiplying constant. Alternatively, the determinant of matrix that includes collinear vectors equals zero.

Given

$$\vec{a} = \begin{bmatrix} 3 \\ \lambda \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Method 1: saying that vectors $\vec{a}$ and $\vec{b}$ *are* collinear is to say that, for example,

$$\vec{a} = k\,\vec{b} \quad \therefore \quad \begin{bmatrix} 3 \\ \lambda \end{bmatrix} = k \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

where k is the multiplying parameter not equal to zero. Then, this system of equations is

$$\left.\begin{array}{ll} 3 & = 2k \\ \lambda & = 6k \end{array}\right\} \quad \therefore \quad k = 3/2 \qquad \text{and if,} \quad \lambda = 6k \quad \therefore \quad \lambda = 6\,\frac{3}{2} = 9$$

resulting in

$$\vec{a} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \quad \therefore \quad \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

i.e., if $\lambda = 9$, they are dependent.

Method 2: given

$$\vec{b} = k\vec{a} \quad \therefore \quad \begin{bmatrix} 2 \\ 6 \end{bmatrix} = k \begin{bmatrix} 3 \\ \lambda \end{bmatrix}$$

determinant is then calculated as

$$\Delta = \left|\vec{a} \;\; \vec{b}\right| = \begin{vmatrix} 3 & 2 \\ \lambda & 6 \end{vmatrix} = 3 \times 6 - 2\lambda = 18 - 2\lambda = 0 \quad \therefore \quad \lambda = 9$$

In conclusion, for any $\lambda \neq 9$ these two vectors are independent (i.e. not parallel). Otherwise for $\lambda = 9$ they are dependent (i.e. parallel), which is the consequence of a simple $3/2$ multiplication factor between the two vectors.

8.53. **Method 1:** given vectors

$$\vec{a} = \begin{bmatrix} 6 \\ 8 \\ 4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix}$$

it is evident that $\vec{a}$ may be factored as

$$\vec{a} = \begin{bmatrix} 6 \\ 8 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \equiv 2\,\vec{b}$$

which is to say that $(\vec{a}, \vec{b})$ are collinear (i.e., dependent) regardless of λ.

Method 2: the determinant of matrix of these three vectors is

$$\Delta = \begin{vmatrix} 6 & 3 & \lambda \\ 8 & 4 & 0 \\ 4 & 2 & 1 \end{vmatrix} \begin{matrix} 6 & 3 \\ 8 & 4 \\ 4 & 2 \end{matrix}$$

$$\begin{aligned} &= +6 \times 4 \times 1 + 3 \times 0 \times 4 + \lambda \times 8 \times 2 \\ &\quad - 4 \times 4 \times \lambda - 2 \times 0 \times 6 - 1 \times 8 \times 3 \\ &= \cancel{24} + \cancel{16\lambda} - \cancel{16\lambda} - \cancel{24} = 0 \end{aligned}$$

that is to say, the determinant equals zero regardless of λ; consequently, these three vectors are independent for any value of λ.

8.6 Vector Space

The modern interpretation of matrices is illustrated by multidimensional space, where column vectors are assembled together into a matrix, which then depicts various transformations of the given space. In order to visualize these transformations, it is sufficient to follow the base vector transformations. Note that base vectors define the "unit volume of their respective space". That is to say, in the case of 1D space, it is "unit" line length; in the case of 2D space, it is "unit" square surface; and in the case of 3D or higher-order space, it is "unit volume". It is assumed that the space is perfectly "stretchable" in any direction and to any distance and direction. A banal example of a 2D space (epitomized as a plain sheet of paper) transformation may be a simple rotation by 90° clockwise relative to one of its corner points; see Sect. 8.3.

8.54. Given matrix–vector product

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

it may be calculated as a linear combination.

$$L\,\vec{v} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

The geometric interpretation of this operation may be illustrated by vector $\vec{v} = 3\vec{i} + 0\vec{j}$ that is transformed to $L(\vec{v}) = 3\,L(\vec{i}) + 0\,L(\vec{j}) \equiv 3\vec{i} + 6\vec{j}$ by means of space transformation L; see Fig. 8.18. Note that coordinates of transformed $L(\vec{i}\,)$ and $L(\vec{j}\,)$ are set by the columns of L transformation matrix.

Therefore, the result of matrix–vector product $L(\vec{v}\,) = (3, 6)$ is shown in the units of the original (i.e., non-transformed basis vectors $\vec{i}$ and $\vec{j}$), as illustrated in Fig. 8.18.

8.55. Given matrix–vector product

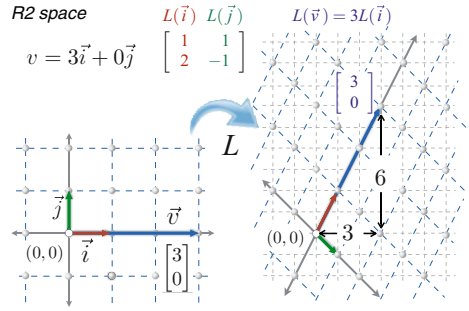


Fig. 8.18 Example P.8.54

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1+\sqrt{3} \\ 2 \end{bmatrix}$$

it may be calculated as a linear combination.

$$L\,\vec{v} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1+\sqrt{3} \\ 2 \end{bmatrix} = (1+\sqrt{3}) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3+\sqrt{3} \\ 2\sqrt{3} \end{bmatrix}$$

Note however that

$$\begin{aligned} 3+\sqrt{3} &= \sqrt{3}\left(\frac{3}{\sqrt{3}}+1\right) = \sqrt{3}\left(\frac{3}{\sqrt{3}}\,\frac{\sqrt{3}}{\sqrt{3}}+1\right) = \sqrt{3}\left(\frac{\not{3}\sqrt{3}}{\not{3}}+1\right) \\ &= \sqrt{3}\left(1+\sqrt{3}\right) \end{aligned}$$

that is to say,

$$L\,\vec{v} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1+\sqrt{3} \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}\,(1+\sqrt{3}) \\ 2\sqrt{3} \end{bmatrix} = \sqrt{3} \begin{bmatrix} 1+\sqrt{3} \\ 2 \end{bmatrix} = \sqrt{3}\,\vec{v}$$

In other words, after the space transformation L, this particular vector $\vec{v}$ stayed *collinear*; it was simply multiplied by a constant "$\sqrt{3}$"; see Fig. 8.19. Compare with the vector $\vec{v}$ in P.8.54 subjected to the same space transformation L.

Reminder: If matrix L does not affect the direction of a certain vector $\vec{v}$, as

$$L\,\vec{v} = \lambda\,\vec{v}$$

then it is said that this particular vector $\vec{v}$ is an *eigenvector* of its associated transformation matrix L and constant λ is referred to as an *eigenvalue* of this matrix L. Existence of eigenvalues and eigenvectors for a given matrix is a very important property. Not all matrices have eigenvalues and subsequently do not have eigenvectors.

Fig. 8.19 Example P.8.55

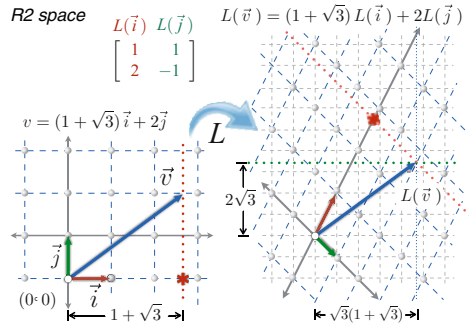
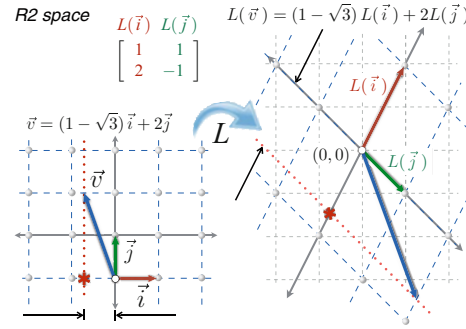


Fig. 8.20 Example P.8.56



8.56. Given matrix–vector product

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1-\sqrt{3} \\ 2 \end{bmatrix}$$

it may be calculated as a linear combination.

$$L\,\vec{v} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1-\sqrt{3} \\ 2 \end{bmatrix} = (1-\sqrt{3}) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3-\sqrt{3} \\ -2\sqrt{3} \end{bmatrix}$$

Note however that

$$\begin{aligned} 3-\sqrt{3} &= -\sqrt{3}\left(-\frac{3}{\sqrt{3}}+1\right) = -\sqrt{3}\left(-\frac{3}{\sqrt{3}}\,\frac{\sqrt{3}}{\sqrt{3}}+1\right) = -\sqrt{3}\left(-\frac{3\sqrt{3}}{3}+1\right) \\ &= -\sqrt{3}\left(1-\sqrt{3}\right) \end{aligned}$$

that is to say,

$$L\,\vec{v} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1-\sqrt{3} \\ 2 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}\,(1-\sqrt{3}) \\ -2\sqrt{3} \end{bmatrix} = -\sqrt{3} \begin{bmatrix} 1-\sqrt{3} \\ 2 \end{bmatrix} = -\sqrt{3}\,\vec{v}$$

In other words, after the space transformation L, this particular vector $\vec{v}$ stayed *collinear*; it was simply multiplied by a constant "$-\sqrt{3}$"; see Fig. 8.20 (compare with P.8.54). In conclusion, this particular constant $\lambda = -\sqrt{3}$ happens to be an *eigenvalue*, and therefore, this particular vector $\vec{v}$ is its associated *eigenvector* of the given transformation matrix L.

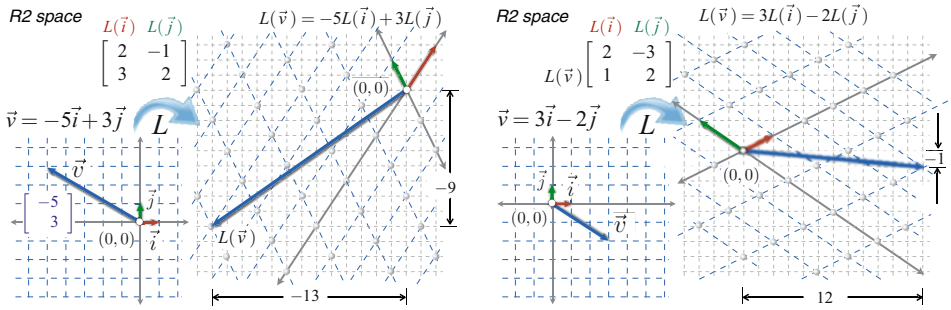


Fig. 8.21 Example P.8.57, P.8.58

8.57. Given matrix–vector product

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

it may be calculated as a linear combination.

$$L\,\vec{v} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \end{bmatrix} = -5 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -13 \\ -9 \end{bmatrix}$$

That is, vector $\vec{v} = -5\vec{i} + 3\vec{j}$ is transformed to $L(\vec{v}) = -5\,L(\vec{i}) + 3\,L(\vec{j}) \equiv -13\vec{i} - 9\vec{j}$ by means of space transformation L; see Fig. 8.21. Note that coordinates of transformed $L(\vec{i}\,)$ and $L(\vec{j}\,)$ are set by the columns of L transformation matrix.

Therefore, the result of matrix–vector product $L(\vec{v}\,) = (-13, -9)$ is shown in the units of the original (i.e., non-transformed basis vectors $\vec{i}$ and $\vec{j}$), as illustrated in Fig. 8.21 (left).

8.58. Given matrix–vector product

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

it may be calculated as a linear combination.

$$L\,\vec{v} = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \end{bmatrix}$$

That is, vector $\vec{v} = 3\vec{i} - 2\vec{j}$ is transformed to $L(\vec{v}) = 3\,L(\vec{i}) - 2\,L(\vec{j}) \equiv 12\vec{i} - \vec{j}$ by means of space transformation L; see Fig. 8.21 (right). Note that coordinates of transformed $L(\vec{i}\,)$ and $L(\vec{j}\,)$ are set by the columns of L transformation matrix.

Therefore, the result of matrix–vector product $L(\vec{v}\,) = (12, -1)$ is shown in the units of the original (i.e., non-transformed basis vectors $\vec{i}$ and $\vec{j}$), as illustrated in Fig. 8.21 (right).

8.59. Given matrix–vector product

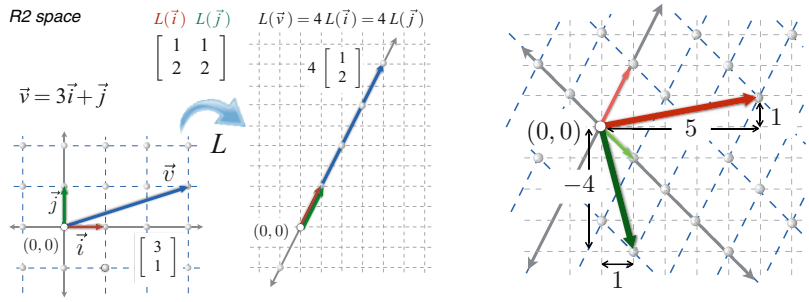


Fig. 8.22 Example P.8.59, P.8.60

$$\begin{bmatrix}1&1\\2&2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}$$

it may be calculated as a linear combination.

$$L\,\vec{v}=\begin{bmatrix}1&1\\2&2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=3\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}4\\8\end{bmatrix}=4\begin{bmatrix}1\\2\end{bmatrix}=4\,L(\vec{i}\,)=4\,L(\vec{j}\,)$$

It is to be noted that the two rows of transformation matrix are *not independent*, that is to say, one can be derived as the multiple of the other.

Consequently, $L(\vec{i}\,)$ and $L(\vec{j}\,)$ are collinear, and furthermore, the transformed vector $L(v)$ is aligned with both of them at the same time. Another hint is that determinant $|L|=0$. The geometrical interpretation of this case is that the original 2D space collapsed into 1D space; see Fig. 8.22 (left). This is the general consequence of transformations whose determinant equals zero.

8.60. Given matrix–matrix product

$$\begin{bmatrix}1&1\\2&-1\end{bmatrix}\begin{bmatrix}2&-1\\3&2\end{bmatrix}$$

geometrically, it may be interpreted as "linear transformation of linear transformation", i.e., two transformations happening at the same time. That is to say, in practical sense, matrix–matrix product may be decomposed into two matrix–vector products. Then, the two resulting vectors are simply combined in the final matrix as

$$\begin{bmatrix}1&1\\2&-1\end{bmatrix}\begin{bmatrix}2&-1\\3&2\end{bmatrix}=\begin{bmatrix}i_x&j_x\\i_y&j_y\end{bmatrix}$$

That is to say,

$$\vec{i}:\quad \begin{bmatrix}1&1\\2&-1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=2\begin{bmatrix}1\\2\end{bmatrix}+3\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}(2+3)\\(4-3)\end{bmatrix}=\begin{bmatrix}5\\1\end{bmatrix}$$

$$\vec{j}:\quad \begin{bmatrix}1&1\\2&-1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}=(-1)\begin{bmatrix}1\\2\end{bmatrix}+2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}(-1+2)\\(-2-2)\end{bmatrix}=\begin{bmatrix}1\\-4\end{bmatrix}$$

so that the final result is

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & -4 \end{bmatrix}$$

The overall transformation is illustrated in Fig. 8.22 (right) where narrow line arrows represent the first transformations of $\vec{i}$ and $\vec{j}$ at $(1, 2)$ and $(1, -1)$ (i.e., left-side matrix), as measured by the units of the original space. Then, wide line arrows represent the final destination of $\vec{i}$ and $\vec{j}$ after the second transformations (i.e., right-side matrix).

8.61. Given matrix–matrix product

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

then, it follows

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} i_x & j_x \\ i_y & j_y \end{bmatrix}$$

where

$$\vec{i}: \quad \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (4-1) \\ (6+2) \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\vec{j}: \quad \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = (-3) \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (-6-2) \\ (-9+4) \end{bmatrix} = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ 8 & -5 \end{bmatrix}$$

8.62. Given matrix–matrix product

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

it follows that

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} i_x & j_x \\ i_y & j_y \end{bmatrix}$$

where

$$\vec{i}: \quad \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} (2-6) \\ (1+4) \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

$$\vec{j}: \quad \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} (2+3) \\ (1-2) \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ 5 & -1 \end{bmatrix}$$

8.63. Given matrix–matrix product

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

it follows that

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} i_x & j_x \\ i_y & j_y \end{bmatrix}$$

where

$$\vec{i}: \quad \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (2-2) \\ (3+4) \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$$

$$\vec{j}: \quad \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (2+1) \\ (3-2) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 7 & 1 \end{bmatrix}$$

By comparison with P.8.60, note that matrix–matrix multiplication is *not* a commutative operation.

8.64. Given matrix–matrix product

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

it follows that

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} i_x & j_x \\ i_y & j_y \end{bmatrix}$$

where

$$\vec{i}: \quad \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} (4-9) \\ (2+6) \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$$

$$\vec{j}: \quad \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = (-1) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} (-2-6) \\ (-1+4) \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -5 & -8 \\ 8 & 3 \end{bmatrix}$$

By comparison with P.8.61, note that matrix–matrix multiplication is *not* a commutative operation.

8.65. Given matrix–matrix product

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

it follows that

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} i_x & j_x \\ i_y & j_y \end{bmatrix}$$

where

$$\vec{i}: \quad \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} (2+1) \\ (4-1) \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\vec{j}: \quad \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = (-3) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} (-3+2) \\ (-6-2) \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 3 & -8 \end{bmatrix}$$

By comparison with P.8.65, note that matrix–matrix multiplication is *not* a commutative operation.

8.66. Two non-square matrices are multiplied as same as two square matrices while keeping in mind that

Reminder: product C of two non-square matrices A, B is possible if the *number of columns* n in the left-side matrix is equal to the *number or rows* k in the matrix on the right side. The resulting matrix has the number of rows m of the left-side matrix and the number of columns l of the right-side matrix, as

$$A_{m,n}\, B_{n,l} = C_{m,l}$$

which is not necessarily a square matrix.

Given matrix–product of two vectors

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

note that the left-side matrix is in "1, 3" format and the right-side matrix is in "3, 1" format; thus, the resulting product must be a matrix sized as "1, 1", because the number of rows on the left is "1" and the number of columns on the right-side vector is also "1". Therefore,

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$$

8.67. Given matrix–product of two vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

note that the left-side matrix is in "3, 1" format and the right-side matrix is in "1, 3" format; thus, the resulting product must be a matrix sized as "3, 3", because the number of rows on the left is "3" and the number of columns on the right-side vector is also "3". Therefore,

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 & 1 \cdot 2 & 1 \cdot 3 \\ 2 \cdot 1 & 2 \cdot 2 & 2 \cdot 3 \\ 3 \cdot 1 & 3 \cdot 2 & 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

8.68. Given matrix–product of two vectors

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 0 & 1 \end{bmatrix}$$

note that the left-side matrix is in "2, 3" format and the right-side matrix is in "3, 2" format; thus, the resulting product must be a matrix sized as "2, 2", because the number of rows on the left is "2" and the number of columns on the right-side vector is also "2". Therefore,

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} i_x & j_x \\ i_y & j_y \end{bmatrix}$$

where

$$\vec{i}: \quad \begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

$$\vec{j}: \quad \begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ -7 & 1 \end{bmatrix}$$

8.69. Given matrix–product of two vectors

$$\begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \end{bmatrix}$$

note that the left-side matrix is in "3, 2" format and the right-side matrix is in "2, 3" format; thus, the resulting product must be a matrix sized as "3, 3", because the number of rows on the left is "3" and the number of columns on the right-side vector is also "3". Therefore,

$$\begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} i_x & j_x & k_x \\ i_y & j_y & k_y \\ i_z & j_z & k_z \end{bmatrix}$$

where

$$\vec{i}: \quad \begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 17 \\ -2 \end{bmatrix}$$

$$\vec{j}: \quad \begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{k}: \quad \begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 4 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 9 \\ 17 & 1 & 1 \\ -2 & -1 & 4 \end{bmatrix}$$

8.7 Eigenvalues and Eigenvectors

8.70. Simple calculation of a matrix–vector product is

$$A\ \vec{v} = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} (3 \times 3 - 1 \times 3) \\ (2 \times 3 - 4 \times 1) \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \lambda\ \vec{v}$$

Therefore, $\vec{v}$ is an eigenvector of matrix A, and its associated eigenvalue is $\lambda = 2$.

8.71. Simple calculation of a matrix–vector product is

$$A\,\vec{v} = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (2\times 3 - 1\times 3) \\ (2\times 2 - 4\times 1) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Therefore, $\vec{v}$ is *not* an eigenvector of matrix A because there is no eigenvalue λ for which the definition equation is possible.

8.72. Given

$$\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$$

one possible procedure to calculate its eigenvalues and eigenvectors (if existing) may be:

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\det(A - \lambda I) = 0$$

$$\therefore$$

$$\det(A - \lambda I) = \left| \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$= \begin{vmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda) - 2(-1) = \lambda^2 - 9\lambda + 20$$

$$= \lambda^2 - 5\lambda - 4\lambda + 20 = \lambda(\lambda - 5) - 4(\lambda - 5) = (\lambda - 5)(\lambda - 4) = 0$$

$$\therefore \quad \underline{\lambda_1 = 4,\ \lambda_2 = 5}$$

   Therefore, two distinct eigenvalues of matrix A are $\lambda_1 = 4,$ and $\lambda_2 = 5$.
2. *Eigenvectors* $\vec{v} = (x, y)$ are calculated for each eigenvalue $\lambda_{1,2}$ so that

$$(A - \lambda_i I)\ \vec{v}_i = \vec{0}\ \ (i = 1, 2)$$

   Case $\lambda_1 = 4$:

$$(A - \lambda_1 I)\ \vec{v}_1 = \left(\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6-4 & -1 \\ 2 & 3-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Therefore, the objective is to calculate (x, y) coordinates of $\vec{v}_1$. Evidently, rows of this simple 2D matrix are not independent, which implies that this system of equations must have a parametrized set of solutions. It may be resolved in the extended matrix form as

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \leftarrow (1) \div 2 = \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 2 & -1 & 0 \end{array}\right] \leftarrow (2) - 2(1)$$

$$= \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow (1) \\ \Rightarrow \quad 0x + 0y = 0 \quad !!! \end{array}$$

That is to say, as the consequence of two equations being dependent, the second equation collapses into its trivial case. Thus, because it is multiplied by zero, coordinate y may take an arbitrary value (except the infinity), and the second equation is still correct. That being the case, it may be parametrized as $y = t$, where t is an arbitrary parameter. By substitution of y in $\leftarrow (1)$, it follows that

$$1\,x - \frac{1}{2}\,y = 0 \quad \therefore \quad 1\,x - \frac{1}{2}\,t = 0 \quad \Rightarrow x = \frac{1}{2}\,t$$

which is to say, the first equation is correct for any value of y as long as y is two times the value of x. It means that the parametrized form of $\vec{v}_1$ is

$$\vec{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t/2 \\ t \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 2$ as a convenient way to derive integer coordinates.

Case $\lambda_2 = 5$: repeated procedure results in

$$(A - \lambda_2\, I)\ \vec{v}_2 = \left[\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6-5 & -1 \\ 2 & 3-5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore,

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \leftarrow (2) - 2(1) \end{array} = \qquad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow 0x + 0y = 0 \quad \therefore \quad y = t$$

and

$$1\,x - 1\,y = 0 \quad \therefore \quad x - 1\,t = 0 \quad \Rightarrow x = t$$

which is to say that y must be equal to x and therefore the parametrized form of $\vec{v}_2$ is

$$\vec{v}_2 = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

3. *Verification:*

$$A\,\vec{v}_1 = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} (1 \cdot 6 + 2 \cdot (-1)) \\ (1 \cdot 2 + 2 \cdot 3) \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda_1\,\vec{v}_1$$

$$A\,\vec{v}_2 = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1\cdot 6 + 1\cdot(-1)) \\ (1\cdot 2 + 1\cdot 3) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_2\,\vec{v}_2$$

8.73. Given

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$$

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\det(A - \lambda I) = 0$$

$$\therefore$$

$$\begin{aligned} \det(A - \lambda I) &= \left|\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = \left|\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| \\ &= \begin{vmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = (-6-\lambda)(5-\lambda) - 4\cdot 3 = \lambda^2 + \lambda - 42 \\ &= \lambda^2 - 6\lambda + 7\lambda - 42 = \lambda(\lambda-6) + 7(\lambda-6) = (\lambda-6)(\lambda+7) = 0 \\ &\therefore\ \underline{\lambda_1 = 6,\ \lambda_2 = -7} \end{aligned}$$

   Therefore, two distinct eigenvalues of matrix A are $\lambda_1 = 6,$ and $\lambda_2 = -7$.
2. *Eigenvectors* $\vec{v} = (x, y)$ are calculated for each eigenvalue $\lambda_{1,2}$ so that

$$(A - \lambda_i I)\ \vec{v}_i = \vec{0}\ \ (i = 1, 2)$$

Case $\lambda_1 = 6$:

$$\begin{aligned} (A - \lambda_1 I)\ \vec{v}_1 &= \left[\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6-6 & 3 \\ 4 & 5-6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} -12 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Therefore, the objective is to calculate (x, y) coordinates of $\vec{v}_1$. Evidently, rows of this simple 2D matrix are not independent, which implies that this system of equations must have a parametrized set of solutions. It may be resolved in the extended matrix form as

$$\left[\begin{array}{cc|c} -12 & 3 & 0 \\ 4 & -1 & 0 \end{array}\right] \leftarrow (1) \div [-12] = \left[\begin{array}{cc|c} 1 & -1/4 & 0 \\ 4 & -1 & 0 \end{array}\right] \leftarrow (2) - 4(1)$$

$$= \left[\begin{array}{cc|c} 1 & -1/4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow\ 0x + 0y = 0 \quad !!!$$

That is to say, as the consequence of two equations being dependent, the second equation collapses into its trivial case. Thus (see A.8.72), $\underline{y = t}$, where t is an arbitrary parameter. By substitution of

y in the first row, it follows that

$$1\,x - \frac{1}{4}\,y = 0 \;\;\therefore\;\; 1\,x - \frac{1}{4}\,t = 0 \;\;\Rightarrow \underline{x = \frac{1}{4}\,t}$$

which is to say that the first equation is correct for any value of y as long as it is four times the value of x. It means that the parametrized form of $\vec{v}_1$ is

$$\vec{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t/4 \\ t \end{bmatrix} = t \begin{bmatrix} 1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 4$ as a convenient way to derive integer coordinates.

Case $\lambda_2 = -7$: repeated procedure results in

$$(A - \lambda_2\, I)\;\vec{v}_2 = \left[\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - (-7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6+7 & 3 \\ 4 & 5+7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore,

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 4 & 12 & 0 \end{array}\right] \leftarrow (2) - 4(1) = \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \;\Rightarrow 0x + 0y = 0 \;\;\therefore\;\; \underline{y = t}$$

and

$$1\,x + 3\,y = 0 \;\;\therefore\;\; x + 3\,t = 0 \;\;\Rightarrow \underline{x = -3t}$$

which is to say, in plain words, that x must be equal to "$-3y$". Therefore, $\vec{v}_2$ is

$$\vec{v}_2 = \begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

3. *Verification:*

$$A\,\vec{v}_1 = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} (1\cdot(-6) + 4\cdot 3) \\ (1\cdot 4 + 5\cdot 4) \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \lambda_1\,\vec{v}_1$$
$$A\,\vec{v}_2 = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} ((-3)\cdot(-6) + 1\cdot 3) \\ ((-3)\cdot 4 + 1\cdot 5) \end{bmatrix} = \begin{bmatrix} 21 \\ -7 \end{bmatrix} = -7\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \lambda_2\,\vec{v}_2$$

8.74. Given

$$\begin{bmatrix} 10 & -5 \\ 5 & 2 \end{bmatrix}$$

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\begin{aligned} &\det(A - \lambda I) = 0 \\ &\qquad \therefore \\ &\det(A - \lambda I) = \left| \begin{bmatrix} 10 & -5 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} 10 & -5 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| \\ &\qquad = \begin{vmatrix} 10-\lambda & -5 \\ 5 & 2-\lambda \end{vmatrix} = (10-\lambda)(2-\lambda) - 5 \cdot (-5) = \lambda^2 - 12\lambda + 45 \\ &\qquad = 0 \\ &\qquad \therefore \ \underline{\lambda_1 = 6 - 3j, \ \lambda_2 = 6 + 3j} \quad (j^2 = -1) \end{aligned}$$

   Therefore, two distinct eigenvalues of matrix A are $\lambda_1 = 6 - 3j$, and $\lambda_2 = 6 + 3j$.
2. *Eigenvectors* $\vec{v} = (x, y)$ are calculated for each eigenvalue $\lambda_{1,2}$ so that

$$(A - \lambda_i I)\ \vec{v}_i = \vec{0} \quad (i = 1, 2)$$

Case $\lambda_1 = 6 - 3j$:

$$\begin{aligned} (A - \lambda_1 I)\ \vec{v}_1 &= \begin{bmatrix} 10 - (6 - 3j) & -5 \\ 5 & 2 - (6 - 3j) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 4 + 3j & -5 \\ 5 & -4 + 3j \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Complex matrices may be solved as same as matrices with real coefficients, as

$$\begin{bmatrix} 4+3j & -5 & 0 \\ 5 & -4+3j & 0 \end{bmatrix} \leftarrow (1) \cdot [4 - 3j] = \begin{bmatrix} 25 & -20+15j & 0 \\ 5 & -4+3j & 0 \end{bmatrix} \leftarrow (1) \div 25$$

$$= \begin{bmatrix} 1 & -4/5 + 3/5\, j & 0 \\ 5 & -4+3j & 0 \end{bmatrix} \leftarrow (2) - 5(1) = \begin{bmatrix} 1 & -4/5 + 3/5\, j & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \quad \Rightarrow \quad 0x + 0y = 0 \quad \therefore \quad \underline{y = t}$$

and

$$1\,x + (-4/5 + 3/5\, j)\, y = 0 \quad \therefore \quad x + (-4/5 + 3/5\, j)\, t = 0 \quad \Rightarrow \underline{x = (4/5 - 3/5\, j)\, t}$$

so that

$$\vec{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (4/5 - 3/5\, j)\, t \\ t \end{bmatrix} = t \begin{bmatrix} 4/5 - 3/5\, j \\ 1 \end{bmatrix} = \begin{bmatrix} 4 - 3\, j \\ 5 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 5$ as a convenient way to derive integer coordinates.

Case $\lambda_2 = 6 + 3j$:

$$(A - \lambda_2 I)\ \vec{v}_2 = \begin{bmatrix} 10 - (6+3j) & -5 \\ 5 & 2 - (6+3j) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 - 3j & -5 \\ 5 & -4 - 3j \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Again,

$$\left[\begin{array}{cc|c} 4 - 3j & -5 & 0 \\ 5 & -4 - 3j & 0 \end{array}\right] \leftarrow (1) \cdot [4 + 3j] = \left[\begin{array}{cc|c} 25 & -20 - 15j & 0 \\ 5 & -4 - 3j & 0 \end{array}\right] \leftarrow (1) \div 25$$

$$= \left[\begin{array}{cc|c} 1 & -4/5 - 3/5\, j & 0 \\ 5 & -4 - 3j & 0 \end{array}\right] \leftarrow (2) - 5(1) = \left[\begin{array}{cc|c} 1 & -4/5 - 3/5\, j & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$= \quad \Rightarrow \quad 0x + 0y = 0 \quad \therefore \quad \underline{y = t}$$

and

$$1\,x + (-4/5 - 3/5\, j)\, y = 0 \quad \therefore \quad x + (-4/5 - 3/5\, j)\, t = 0 \quad \Rightarrow \underline{x = (4/5 + 3/5\, j)\, t}$$

so that

$$\vec{v}_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (4/5 + 3/5\, j)\, t \\ t \end{bmatrix} = t \begin{bmatrix} 4/5 + 3/5\, j \\ 1 \end{bmatrix} = \begin{bmatrix} 4 + 3\, j \\ 5 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 5$ as a convenient way to derive integer coordinates.

8.75. Given

$$\begin{bmatrix} 2 & -5 \\ 4 & 6 \end{bmatrix}$$

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\det(A - \lambda I) = 0$$

$$\therefore$$

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -5 \\ 4 & 6 - \lambda \end{vmatrix} = (2 - \lambda)(6 - \lambda) - 4 \cdot (-5) = \lambda^2 - 8\lambda + 32 = 0$$

$$\therefore \quad \underline{\lambda_1 = 4 - 4j,\ \lambda_2 = 4 + 4j} \quad (j^2 = -1)$$

Therefore, two distinct eigenvalues of matrix A are $\lambda_1 = 4 - 4j$, and $\lambda_2 = 4 + 4j$.

2. *Eigenvectors* $\vec{v} = (x, y)$ are calculated for each eigenvalue $\lambda_{1,2}$ so that

$$(A - \lambda_i\, I)\ \vec{v}_i = \vec{0}\ \ (i = 1, 2)$$

Case $\lambda_1 = 4 - 4j$:

$$(A - \lambda_1\, I)\ \vec{v}_1 = \begin{bmatrix} 2-(4-4j) & -5 \\ 4 & 6-(4-4j) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2+4j & -5 \\ 4 & 2+4j \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Complex matrices may be solved as same as matrices with real coefficients, as

$$\left[\begin{array}{cc|c} -2+4j & -5 & 0 \\ 4 & 2+4j & 0 \end{array}\right] \leftarrow (1)\cdot[-2-4j] = \left[\begin{array}{cc|c} 20 & 10+20j & 0 \\ 4 & 2+4j & 0 \end{array}\right] \leftarrow (1) \div 20$$

$$= \left[\begin{array}{cc|c} 1 & 1/2+j & 0 \\ 4 & 2+4j & 0 \end{array}\right] \leftarrow (2) - 4(1) = \left[\begin{array}{cc|c} 1 & 1/2+j & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$= \ \Rightarrow\ 0x + 0y = 0\ \ \therefore\ \ \underline{y = t}$$

and

$$1\,x + (1/2 + j)\,y = 0\ \ \therefore\ \ x + (1/2 + j)\,t = 0\ \ \Rightarrow \underline{x = (-1/2 - j)\,t}$$

so that

$$\vec{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (-1/2 - j)\,t \\ t \end{bmatrix} = t \begin{bmatrix} -1/2 - j \\ 1 \end{bmatrix} = \begin{bmatrix} -1 - 2\,j \\ 2 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 2$ as a convenient way to derive integer coordinates.

Case $\lambda_2 = 4 + 4j$:

$$(A - \lambda_2\, I)\ \vec{v}_2 = \begin{bmatrix} 2-(4+4j) & -5 \\ 4 & 6-(4+4j) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2-4j & -5 \\ 4 & 2-4j \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Complex matrices may be solved as same as matrices with real coefficients, as

$$\left[\begin{array}{cc|c} -2-4j & -5 & 0 \\ 4 & 2-4j & 0 \end{array}\right] \leftarrow (1)\cdot[-2+4j] = \left[\begin{array}{cc|c} 20 & 10-20j & 0 \\ 4 & 2-4j & 0 \end{array}\right] \leftarrow (1) \div 20$$

$$= \left[\begin{array}{cc|c} 1 & 1/2-j & 0 \\ 4 & 2-4j & 0 \end{array}\right] \leftarrow (2) - 4(1) = \left[\begin{array}{cc|c} 1 & 1/2-j & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$= \ \Rightarrow\ 0x + 0y = 0\ \ \therefore\ \ \underline{y = t}$$

and

$$1\,x + (1/2 - j)\,y = 0\ \ \therefore\ \ x + (1/2 - j)\,t = 0\ \ \Rightarrow \underline{x = (-1/2 + j)\,t}$$

so that

$$\vec{v}_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (-1/2 + j)\, t \\ t \end{bmatrix} = t \begin{bmatrix} -1/2 + j \\ 1 \end{bmatrix} = \begin{bmatrix} -1 + 2\, j \\ 2 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 2$ as a convenient way to derive integer coordinates.

8.76. Given

$$\begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\det (A - \lambda\, I) = 0$$

$$\therefore$$

$$\det (A - \lambda\, I) = \begin{vmatrix} \sqrt{3}/2 - \lambda & -1/2 \\ 1/2 & \sqrt{3}/2 - \lambda \end{vmatrix} = (\sqrt{3}/2 - \lambda)(\sqrt{3}/2 - \lambda) + 1/4 = \lambda^2 - \sqrt{3}\,\lambda + 1 = 0$$

$$\therefore \quad \underline{\lambda_1 = \sqrt{3}/2 - 1/2\, j, \quad \lambda_2 = \sqrt{3}/2 + 1/2\, j} \quad (j^2 = -1)$$

Therefore, two distinct eigenvalues of matrix A are $\lambda_1 = \sqrt{3}/2 - 1/2\, j$, and $\lambda_2 = \sqrt{3}/2 + 1/2\, j$.

2. *Eigenvectors* $\vec{v} = (x, y)$ are calculated for each eigenvalue $\lambda_{1,2}$ so that

$$(A - \lambda_i\, I)\ \vec{v_i} = \vec{0} \quad (i = 1, 2)$$

Case $\lambda_1 = \sqrt{3}/2 - 1/2\, j$:

$$(A - \lambda_1\, I)\ \vec{v}_1 = \begin{bmatrix} \sqrt{3}/2 - (\sqrt{3}/2 - 1/2\, j) & -1/2 \\ 1/2 & \sqrt{3}/2 - (\sqrt{3}/2 - 1/2\, j) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1/2\, j & -1/2 \\ 1/2 & 1/2\, j \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Complex matrices may be solved as same as matrices with real coefficients, as

$$\left[\begin{array}{cc|c} 1/2\, j & -1/2 & 0 \\ 1/2 & 1/2\, j & 0 \end{array}\right] \leftarrow (1) \cdot [-2\, j] = \left[\begin{array}{cc|c} 1 & j & 0 \\ 1/2 & 1/2\, j & 0 \end{array}\right] \leftarrow (2) - 1/2(1)$$

$$= \left[\begin{array}{cc|c} 1 & j & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad 0x + 0y = 0 \quad \therefore \quad \underline{y = t}$$

$$1\, x + j\, y = 0 \quad \therefore \quad x + j\, t = 0 \quad \Rightarrow \underline{x = -j\, t}$$

$$\vec{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -j\,t \\ t \end{bmatrix} = t \begin{bmatrix} -j \\ 1 \end{bmatrix} = \begin{bmatrix} -j \\ 1 \end{bmatrix}$$

after the value of parameter t is conveniently set to $t = 1$.

Case $\lambda_2 = \sqrt{3}/2 + 1/2\,j$:

$$(A - \lambda_2 I)\ \vec{v}_2 = \begin{bmatrix} \sqrt{3}/2 - (\sqrt{3}/2 + 1/2\,j) & -1/2 \\ 1/2 & \sqrt{3}/2 - (\sqrt{3}/2 + 1/2\,j) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} -1/2\,j & -1/2 \\ 1/2 & -1/2\,j \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Complex matrices may be solved as same as matrices with real coefficients, as

$$\left[\begin{array}{cc|c} -1/2\,j & -1/2 & 0 \\ 1/2 & -1/2\,j & 0 \end{array}\right] \leftarrow (1) \cdot [2\,j] = \left[\begin{array}{cc|c} 1 & -j & 0 \\ 1/2 & -1/2\,j & 0 \end{array}\right] \leftarrow (2) - 1/2(1)$$
$$= \left[\begin{array}{cc|c} 1 & -j & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad 0x + 0y = 0 \quad \therefore \quad \underline{y = t}$$

and

$$1\,x - j\,y = 0 \quad \therefore \quad x - j\,t = 0 \quad \Rightarrow \underline{x = j\,t}$$

so that

$$\vec{v}_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} j\,t \\ t \end{bmatrix} = t \begin{bmatrix} j \\ 1 \end{bmatrix} = \begin{bmatrix} j \\ 1 \end{bmatrix}$$

after the value of parameter t is conveniently set to $t = 1$.

8.77. Given

$$A = \begin{bmatrix} -2 & -9 & -1 \\ 2 & -6 & -5 \\ -4 & -3 & 4 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

The matrix vector products is

$$A\,\vec{v} = \begin{bmatrix} -2 & -9 & -1 \\ 2 & -6 & -5 \\ -4 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} ((-2)\cdot 2 + (-9)\cdot 1 + (-1)\cdot 1) \\ (2\cdot 2 + (-6)\cdot 1 + (-5)\cdot 1) \\ ((-4)\cdot 2 + (-3)\cdot 1 + 4\cdot 1) \end{bmatrix} = \begin{bmatrix} -14 \\ -7 \\ -7 \end{bmatrix}$$
$$= -7 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \lambda\,\vec{v}$$

Therefore, $\vec{v}$ is an eigenvector and $\lambda = -7$ is an eigenvalue of A.

8.78. Given

$$A = \begin{bmatrix} -2 & -9 & -1 \\ 2 & -6 & -5 \\ -4 & -3 & 4 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$$

The matrix vector products is

$$A\,\vec{v} = \begin{bmatrix} -2 & -9 & -1 \\ 2 & -6 & -5 \\ -4 & -3 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} ((-2)\cdot 4 + (-9)\cdot(-3) + (-1)\cdot 7) \\ (2\cdot 4 + (-6)\cdot(-3) + (-5)\cdot 7) \\ ((-4)\cdot 4 + (-3)\cdot(-3) + 4\cdot 7) \end{bmatrix} = \begin{bmatrix} 12 \\ -9 \\ 21 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix} = \lambda\,\vec{v}$$

Therefore, $\vec{v}$ is an eigenvector and $\lambda = 3$ is an eigenvalue of A.

8.79. Given

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 5-\lambda & 1 \\ 0 & 4 & 2-\lambda \end{vmatrix} \quad \text{(take advantage of zeros to calculate } \Delta\text{)}$$

$$= (3-\lambda)\begin{vmatrix} 5-\lambda & 1 \\ 4 & 2-\lambda \end{vmatrix} - 0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + 0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$$

$$= (3-\lambda)\big((5-\lambda)(2-\lambda) - 4\big)$$

$$= \underline{-\lambda^3 + 10\lambda^2 - 27\lambda + 18 = 0}$$

Three roots of this characteristic polynomial are $\lambda_1 = 1, \lambda_2 = 3,$ and $\lambda_3 = 6$.

2. *Eigenvectors* $\vec{v} = (x, y)$ are calculated for each eigenvalue $\lambda_{1,2,3}$ so that

$$(A - \lambda_i I)\,\vec{v}_i = \vec{0} \quad (i = 1, 2, 3)$$

Case $\lambda_1 = 1$:

$$(A - \lambda_1 I)\,\vec{v}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right] \begin{array}{l} \leftarrow (1) \div [2] \\ \leftarrow (2) \div [4] \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right] \leftarrow (3) - 4(2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \quad 0x + 0y + 0z = 0 \quad \therefore \quad \underline{z = t}$$

and, by substituting z in rows (1) and (2),

$$\begin{aligned} 1\,x + 0\,y + 0\,z = 0 \quad &\Rightarrow x = 0 \\ 0\,x + 1\,y + 1/4\,z = 0 \quad &\Rightarrow y = -1/4\,t \end{aligned}$$

so that

$$\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1/4\,t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 4$ as a convenient way to derive integer coordinates.

Case $\lambda_2 = 3$:

$$(A - \lambda_2\, I)\ \vec{v}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & -1 & 0 \end{array}\right] \begin{array}{l} \Rightarrow \quad 0x + 0y + 0z = 0 \quad \therefore \quad \underline{x = t} \\ \leftarrow (2) \div [2] \\ \leftarrow (3) \div [4] \end{array}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 1 & -1/4 & 0 \end{array}\right] \quad x = t$$

that is to say, $x = t$; then from rows (2) and (3), it follows that

$$\begin{aligned} 0\,x + 1\,y + 1/2\,z = 0 \quad &\Rightarrow y = 1/2\,z \\ 0\,x + 1\,y - 1/4\,z = 0 \quad &\Rightarrow y = -1/4\,z \end{aligned}$$

which is possible only if $z = 0$ followed by $y = 0$. Consequently,

$$\vec{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

after the value of parameter t is conveniently set to $t = 1$.

Case $\lambda_3 = 6$:

$$(A - \lambda_3 I)\ \vec{v}_3 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right] \begin{array}{l} \leftarrow (1) \div [-3] \\ \leftarrow (2) \div [-1] \\ \\ \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow (3) - 4(2) \end{array} =$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad 0x + 0y + 0z = 0 \quad \therefore \quad \underline{z = t}$$

that is to say, $z = t$; then from rows (1) and (2), it follows that

$$1\,x + 0\,y + 0\,z = 0 \quad \Rightarrow \underline{x = 0}$$
$$0\,x + 1\,y - 1\,z = 0 \quad \Rightarrow \underline{y = z = t}$$

In other words, as long as $y = z$, any arbitrary value (except infinity) of t is valid. Consequently,

$$\vec{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

after the value of parameter t is conveniently set to $t = 1$.

3. *Verification:*

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 0 + 0 \cdot (-1) + 0 \cdot 4 \\ 0 \cdot 0 + 5 \cdot (-1) + 1 \cdot 4 \\ 0 \cdot 0 + 4 \cdot (-1) + 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \lambda_1\, \vec{v}_1$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + 5 \cdot 0 + 1 \cdot 0 \\ 0 \cdot 1 + 4 \cdot 0 + 2 \cdot 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_2\, \vec{v}_2$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 \\ 0 \cdot 0 + 5 \cdot 1 + 1 \cdot 1 \\ 0 \cdot 0 + 4 \cdot 1 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \lambda_3\, \vec{v}_3$$

8.80. Given

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{bmatrix}$$

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 5 \\ 0 & 4 & 3-\lambda \end{vmatrix} \quad \text{(take advantage of zeros to calculate } \Delta) \\ &= (2-\lambda)\begin{vmatrix} 4-\lambda & 5 \\ 4 & 3-\lambda \end{vmatrix} - 0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + 0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} \\ &= (2-\lambda)\big((4-\lambda)(3-\lambda) - 20\big) \\ &= \underline{-\lambda^3 + 9\lambda^2 - 6\lambda - 16 = 0} \end{aligned}$$

Three roots (see chapters on algebra) of this characteristic polynomial are $\lambda_1 = -1, \lambda_2 = 2,$ and $\lambda_3 = 8$.

2. *Eigenvectors* $\vec{v} = (x, y)$ are calculated for each eigenvalue $\lambda_{1,2,3}$ so that

$$(A - \lambda_i I)\ \vec{v}_i = \vec{0} \quad (i = 1, 2, 3)$$

Case $\lambda_1 = -1$:

$$(A - \lambda_1 I)\ \vec{v}_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 5 \\ 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right] \begin{array}{l} \leftarrow (1) \div [3] \\ \leftarrow (2) \div [5] \\ \\ \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right] \leftarrow (3) - 4(2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow \quad \underline{x = 0} \\ \\ \Rightarrow \quad 0x + 0y + 0z = 0 \quad \therefore \quad \underline{z = t} \end{array}$$

and, by substituting $z = t$ in row (2),

$$0\,x + 1\,y + 1\,z = 0 \quad \Rightarrow y = -t$$

so that

$$\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

Case $\lambda_2 = 2$:

$$(A - \lambda_2 I)\ \vec{v}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right] \begin{array}{l} \Rightarrow \ x = t \\ \leftarrow (2) \div [2] \\ \leftarrow (3) \div [4] \end{array} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 5/2 & 0 \\ 0 & 1 & 1/4 & 0 \end{array}\right]$$

which is to say that $x = t$, and from rows (2) and (3),

$$0x + 1y + 5/2\, z = 0 \Rightarrow y = -5/2\, z$$
$$0x + 1y + 1/4\, z = 0 \Rightarrow y = -1/4\, z$$

which is possible only if $z = 0$; then it follows that $y = 0$. That being the case,

$$\vec{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

Case $\lambda_3 = 8$:

$$(A - \lambda_3 I)\ \vec{v}_3 = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -4 & 5 \\ 0 & 4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} -6 & 0 & 0 & 0 \\ 0 & -4 & 5 & 0 \\ 0 & 4 & -5 & 0 \end{array}\right] \begin{array}{l} \leftarrow (1) \div [-6] \\ \leftarrow (2) \div [-4] \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -5/4 & 0 \\ 0 & 4 & -5 & 0 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow (3) - 4(2) \end{array} =$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -5/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow \ x = 0 \\ \\ \Rightarrow \ 0x + 0y + 0z = 0 \ \ \therefore \ z = t \end{array}$$

which is to say that $z = t$, and from row (2),

$$0\,x + 1\,y - {}^5\!/_4\,z = 0 \Rightarrow y = {}^5\!/_4\,t$$

That being the case,

$$\vec{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ {}^5\!/_4\,t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ {}^5\!/_4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 4$ as a convenient way to derive integer coordinates.

3. *Verification:*

$$\begin{bmatrix} 2&0&0 \\ 0&4&5 \\ 0&4&3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 0 + 0\cdot(-1) + 0\cdot 1 \\ 0\cdot 0 + 4\cdot(-1) + 5\cdot 1 \\ 0\cdot 0 + 4\cdot(-1) + 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \lambda_1\,\vec{v}_1$$

$$\begin{bmatrix} 2&0&0 \\ 0&4&5 \\ 0&4&3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 0\cdot 0 + 0\cdot 0 \\ 0\cdot 1 + 4\cdot 0 + 5\cdot 0 \\ 0\cdot 1 + 4\cdot 0 + 3\cdot 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_2\,\vec{v}_2$$

$$\begin{bmatrix} 2&0&0 \\ 0&4&5 \\ 0&4&3 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2\cdot 0 + 0\cdot 5 + 0\cdot 4 \\ 0\cdot 0 + 4\cdot 5 + 5\cdot 4 \\ 0\cdot 0 + 4\cdot 5 + 3\cdot 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 40 \\ 32 \end{bmatrix} = 8 \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} = \lambda_3\,\vec{v}_3$$

8.81. Given

$$\begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix}$$

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 4-\lambda & 6 & 10 \\ 3 & 10-\lambda & 13 \\ -2 & -6 & -8-\lambda \end{vmatrix} \\ &= (4-\lambda)\begin{vmatrix} 10-\lambda & 13 \\ -6 & -8-\lambda \end{vmatrix} - 6\begin{vmatrix} 3 & 13 \\ -2 & -8-\lambda \end{vmatrix} + 10\begin{vmatrix} 3 & 10-\lambda \\ -2 & -6 \end{vmatrix} \\ &= (4-\lambda)\big((10-\lambda)(-8-\lambda) - (-6)\times 13\big) \\ &\quad - 6\big(3(-8-\lambda) - (-2)\times 13\big) \\ &\quad + 10\big(3\times(-6) - (-2)(10-\lambda)\big) \\ &= (-\lambda^3 + 6\lambda^2 - 6\lambda - 8) + (-12 + 18\lambda) + (20 - 20\lambda) \end{aligned}$$

$$= \underline{-\lambda^3 + 6\lambda^2 - 8\lambda = 0}$$

Roots of this characteristic polynomial may be found as

$$-\lambda^3 + 6\lambda^2 - 8\lambda = 0$$
$$-\lambda(\lambda^2 - 6\lambda + 8) = 0$$
$$-\lambda(\lambda^2 - 4\lambda - 2\lambda + 8) = 0$$

$$-\lambda\big(\lambda(\lambda - 4) - 2(\lambda - 4)\big) = 0$$
$$-\lambda(\lambda - 2)(\lambda - 4) = 0$$
$$\therefore$$
$$\lambda_1 = 0, \quad \lambda_2 = 2, \quad \lambda_3 = 4$$

2. *Eigenvectors* $\vec{v} = (x, y)$ are calculated for each eigenvalue $\lambda_{1,2,3}$ so that

$$(A - \lambda_i I)\ \vec{v_i} = \vec{0} \quad (i = 1, 2, 3)$$

Case $\lambda_1 = 0$:

$$(A - \lambda_1 I)\ \vec{v}_1 = A\,\vec{v}_1 = \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 4 & 6 & 10 & 0 \\ 3 & 10 & 13 & 0 \\ -2 & -6 & -8 & 0 \end{array}\right] \leftarrow (1) \div [4] = \left[\begin{array}{ccc|c} 1 & 3/2 & 5/2 & 0 \\ 3 & 10 & 13 & 0 \\ -2 & -6 & -8 & 0 \end{array}\right] \begin{array}{l} \\ \leftarrow (2) - 3 \times (1) \\ \leftarrow (3) \div [-2] \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 3/2 & 5/2 & 0 \\ 0 & 11/2 & 11/2 & 0 \\ 0 & 3/2 & 3/2 & 0 \end{array}\right] \begin{array}{l} \\ \leftarrow (2) \times 2/11 \\ \leftarrow (3) \times 2/3 \end{array} = \left[\begin{array}{ccc|c} 1 & 3/2 & 5/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \leftarrow (3) - (2)$$

$$= \left[\begin{array}{ccc|c} 1 & 3/2 & 5/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow\ 1\,x + 3/2\,y + 3/2\,z = 0 \\ \Rightarrow\ 0\,x + 1\,y + 1\,z = 0 \\ \Rightarrow\ \underline{z = t} \end{array}$$

That is to say, from rows (2) and (1), it follows that

$$y + z = 0 \ \ \therefore\ \ y = -z \ \ \therefore\ \ \underline{y = -t}$$
$$x + \frac{3}{2}y + \frac{5}{2}z = 0 \ \ \therefore\ \ x - \frac{3}{2}t + \frac{5}{2}t = 0 \ \ \therefore\ \ x + t = 0 \ \ \therefore\ \ \underline{x = -t}$$

so that

$$\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

Case $\lambda_2 = 2$:

$$(A - \lambda_2 I)\ \vec{v}_2 = \begin{bmatrix} 2 & 6 & 10 \\ 3 & 8 & 13 \\ -2 & -6 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 2 & 6 & 10 & 0 \\ 3 & 8 & 13 & 0 \\ -2 & -6 & -10 & 0 \end{array}\right] \begin{array}{l} \leftarrow (1) \div [2] \\ \\ \\ \end{array} = \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 3 & 8 & 13 & 0 \\ -2 & -6 & -10 & 0 \end{array}\right] \begin{array}{l} \\ \leftarrow (2) - 3(1) \\ \leftarrow (3) + 2(1) \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow (2) \div [-1] \\ \\ \\ \end{array} = \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow 1x + 3y + 5z = 0 \\ \Rightarrow 0x + 1y + 2z = 0 \\ \Rightarrow \underline{z = t} \end{array}$$

That is to say, from (2) and (1) rows,

$$y + 2z = 0 \ \ \therefore \ \ y = -2z \ \ \therefore \ \ \underline{y = -2t}$$

$$x + 3y + 5z = 0 \ \ \therefore \ \ x - 6t + 5t = 0 \ \ \therefore \ \ x - t = 0 \ \ \therefore \ \ \underline{x = t}$$

and

$$\vec{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

Case $\lambda_3 = 4$:

$$(A - \lambda_3 I)\ \vec{v}_3 = \begin{bmatrix} 0 & 6 & 10 \\ 3 & 6 & 13 \\ -2 & -6 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 0 & 6 & 10 & 0 \\ 3 & 6 & 13 & 0 \\ -2 & -6 & -12 & 0 \end{array}\right] \begin{array}{l} \leftarrow (1) \leftrightarrow (2) \\ \\ \\ \end{array} = \left[\begin{array}{ccc|c} 3 & 6 & 13 & 0 \\ 0 & 6 & 10 & 0 \\ -2 & -6 & -12 & 0 \end{array}\right] \begin{array}{l} \leftarrow (1) \div [3] \\ \\ \leftarrow (3) \div [-2] \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 13/3 & 0 \\ 0 & 6 & 10 & 0 \\ 1 & 3 & 6 & 0 \end{array}\right] \begin{array}{l} \\ \leftarrow (2) \div [6] \\ \leftarrow (3) - (1) \end{array} = \left[\begin{array}{ccc|c} 1 & 2 & 13/3 & 0 \\ 0 & 1 & 5/3 & 0 \\ 0 & 1 & 5/3 & 0 \end{array}\right] \leftarrow (3) - (2)$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 13/3 & 0 \\ 0 & 1 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \Rightarrow 1x + 2y + 13/3\, z = 0 \\ \Rightarrow 0x + 1y + 5/3\, z = 0 \\ \Rightarrow \underline{z = t} \end{array}$$

That is to say, from rows (2) and (1), it follows that

$$y + \frac{5}{3}z = 0 \;\;\therefore\;\; y = -\frac{5}{3}z \;\;\therefore\;\; \underline{y = -\frac{5}{3}t}$$

$$x + 2y + \frac{13}{3}z = 0 \;\;\therefore\;\; x - \frac{10}{3}t + \frac{13}{3}t = 0 \;\;\therefore\;\; x + t = 0 \;\;\therefore\;\; \underline{x = -t}$$

so that

$$\vec{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ -5t/3 \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 3 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 3$ as a convenient way to derive integer coordinates.

3. *Verification:*

$$A\,\vec{v}_1 = \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4\cdot(-1) + 6\cdot(-1) + 10\cdot 1 \\ 3\cdot(-1) + 10\cdot(-1) + 13\cdot 1 \\ -2\cdot(-1) + (-6)\cdot(-1) + (-8)\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= 0 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \lambda_1\,\vec{v}_1 \;\; \text{(i.e., zero times any vector, including } \vec{v}_1)$$

$$A\,\vec{v}_2 = \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4\cdot 1 + 6\cdot(-2) + 10\cdot 1 \\ 3\cdot 1 + 10\cdot(-2) + 13\cdot 1 \\ -2\cdot 1 + (-6)\cdot(-2) + (-8)\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \lambda_2\,\vec{v}_2$$

$$A\,\vec{v}_3 = \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix} \begin{bmatrix} -3 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 4\cdot(-3) + 6\cdot(-5) + 10\cdot 3 \\ 3\cdot(-3) + 10\cdot(-5) + 13\cdot 3 \\ -2\cdot(-3) + (-6)\cdot(-5) + (-8)\cdot 3 \end{bmatrix} = \begin{bmatrix} -12 \\ -20 \\ 12 \end{bmatrix}$$

$$= 4 \begin{bmatrix} -3 \\ -5 \\ 3 \end{bmatrix} = \lambda_3\,\vec{v}_3$$

8.82. Given

$$\begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 2 & -2 \\ 1 & 3-\lambda & -1 \\ -1 & 1 & 1-\lambda \end{vmatrix} \\ &= (2-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ -1 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 3-\lambda \\ -1 & 1 \end{vmatrix} \\ &= (2-\lambda)\big((3-\lambda)(1-\lambda) - (-1)\big) \\ &\quad - 2\big(1(1-\lambda) - 1\big) \\ &\quad - 2\big(1 - (-1)(3-\lambda)\big) \\ &= (-\lambda^3 + 6\lambda^2 - 12\lambda + 8) + 2\lambda + (-8 + 2\lambda) \\ &= \underline{-\lambda^3 + 6\lambda^2 - 8\lambda = 0} \end{aligned}$$

Roots of this characteristic polynomial may be found as (see A.8.81)

$$\lambda_1 = 0, \quad \lambda_2 = 2, \quad \lambda_3 = 4$$

2. *Eigenvectors* $\vec{v} = (x, y)$ are calculated for each eigenvalue $\lambda_{1,2,3}$ so that

$$(A - \lambda_i I)\ \vec{v}_i = \vec{0} \quad (i = 1, 2, 3)$$

Case $\lambda_1 = 0$:

$$(A - \lambda_1 I)\ \vec{v}_1 = A\,\vec{v}_1 = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ 1 & 3 & -1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} (1) \leftrightarrow (2) \\ \\ \leftarrow (3) + (2) \end{array} = \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & 2 & -2 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right] \leftarrow (2) - 2(1)$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right] \leftarrow (3) + (2) = \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned}
&\Rightarrow 1\,x + 3\,y - z = 0\\
&\Rightarrow 0\,x - 4\,y + 0\,z = 0\\
&\Rightarrow \underline{z = t}
\end{aligned}$$

That is to say, from rows (2) and (1), it follows that

$$\begin{aligned}
-4y = 0 \;\;&\therefore\;\; \underline{y = 0}\\
x + 3(0) - t = 0 \;\;&\therefore\;\; \underline{x = t}
\end{aligned}$$

so that

$$\vec{v_1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

Case $\lambda_2 = 2$:

$$(A - \lambda_2 I)\,\vec{v}_2 = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 0 & 2 & -2 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} \\ (1) \leftrightarrow (2) \\ \leftarrow (3) + (2) \end{array} = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \leftarrow (3) - (2)$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow 1\,x + 1\,y - 1\,z = 0 \\ \Rightarrow 0\,x + 2\,y - 2\,z = 0 \\ \Rightarrow \underline{z = t} \end{array}$$

That is to say, from (2) and (1) rows,

$$\begin{aligned}
2y - 2z = 0 \;\;&\therefore\;\; \underline{y = t}\\
x + y - z = 0 \;\;&\therefore\;\; x + t - t = 0 \;\;\therefore\;\; \underline{x = 0}
\end{aligned}$$

and

$$\vec{v_2} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

Case $\lambda_3 = 4$:

$$(A - \lambda_3 I)\ \vec{v}_3 = \begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & -1 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\begin{bmatrix} -2 & 2 & -2 & 0 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & -3 & 0 \end{bmatrix} \begin{matrix} \leftarrow (1) \leftrightarrow (2) \\ \\ \leftarrow (3) + (2) \end{matrix} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -2 & 2 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \begin{matrix} \\ \leftarrow (2) \div [-2] \\ \leftarrow (3) \div [-2] \end{matrix}$$

$$= \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} \Rightarrow 1x - 1y - 1z = 0 \\ \Rightarrow 1x - 1y + 1z = 0 \\ \Rightarrow \underline{z = 0} \end{matrix}$$

That is to say, from rows (1) and (2), it follows that

$$x - y - 0 = 0 \ \ \therefore \ \ \underline{x = y = t}$$
$$x - y + 0 = 0 \ \ \therefore \ \ \underline{x = y = t}$$

so that

$$\vec{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

3. *Verification:*

$$A\,\vec{v}_1 = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 2 \cdot 0 + (-2) \cdot 1 \\ 1 \cdot 1 + 3 \cdot 0 + (-1) \cdot 1 \\ (-1) \cdot 1 + 1 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \lambda_1\,\vec{v}_1$$

(i.e., zero times any vector, including $\vec{v}_1$)

$$A\,\vec{v}_2 = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 0 + 2 \cdot 1 + (-2) \cdot 1 \\ 1 \cdot 0 + 3 \cdot 1 + (-1) \cdot 1 \\ (-1) \cdot 0 + 1 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \lambda_2\,\vec{v}_2$$

$$A\,\vec{v}_3 = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 2 \cdot 1 + (-2) \cdot 0 \\ 1 \cdot 1 + 3 \cdot 1 + (-1) \cdot 0 \\ (-1) \cdot 1 + 1 \cdot 1 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \lambda_3\,\vec{v}_3$$

8.83. Given

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

it follows that

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\det(A - \lambda I) = 0$$

$$\therefore$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 2 \cdot 0 = 0$$

$$\therefore \quad \underline{\lambda_1 = 1, \ \lambda_2 = 1}$$

   That is to say, there is *duplicity* of eigenvalues $\lambda_1 = \lambda_2 = 1$.
2. *Eigenvectors* $\vec{v} = (x, y)$ may be calculated as

$$(A - \lambda_i I) \ \vec{v_i} = \vec{0} \ \ (i = 1, 2)$$

   Case $\lambda_1 = \lambda_2 = 1$:

$$(A - \lambda_1 I) \ \vec{v}_1 = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right] \begin{array}{l} (1) \leftrightarrow (2) \\ \leftarrow (2) \div [2] \end{array} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \ x + 0 \cdot y = 0 \ \ \therefore \ \ \underline{x = 0} \\ \Rightarrow \underline{y = t}$$

   that is to say, all vectors are of the form

$$\vec{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates. Since all eigenvectors are generated by a single eigenvector (by changing t), it is said that the eigenspace has dimension "1".
3. *Verification:*

$$A \ \vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 0 \cdot 1 \\ 2 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda_1 \vec{v}_1$$

8.84. Given

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

it follows that

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\det(A - \lambda I) = 0$$

$$\therefore$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 0 \cdot 0 = 0$$

$$\therefore \ \underline{\lambda_1 = 2, \ \lambda_2 = 2}$$

That is to say, there is *duplicity* of eigenvalues $\lambda_1 = \lambda_2 = 2$.

2. *Eigenvectors* $\vec{v} = (x, y)$ may be calculated as

$$(A - \lambda_i I) \ \vec{v_i} = \vec{0} \ \ (i = 1, 2)$$

Case $\lambda_1 = \lambda_2 = 2$:

$$(A - \lambda_1 I) \ \vec{v}_1 = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow \ 0x + 0y = 0 \\ \Rightarrow 0x + \ 0y = 0 \end{array}$$

Consequently, both coordinates (x, y) may be arbitrary, that is to say, two possible eigenvectors are as follows:

$$\underline{x = t} \ \Rightarrow y = 0 \ \ \therefore \ \ \vec{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underline{y = t} \ \Rightarrow x = 0 \ \ \therefore \ \ \vec{v}_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$. Since there are two distinct eigenvectors, it is said that the eigenspace has dimension "2".

3. *Verification:*

$$A \vec{v}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 0 \cdot 0 \\ 0 \cdot 1 + 2 \cdot 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1 \vec{v}_1$$

$$A \vec{v}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 0 + 0 \cdot 1 \\ 0 \cdot 0 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda_2 \vec{v}_2$$

8.85. Given

$$\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

it follows that

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\det(A - \lambda I) = 0$$

$$\therefore$$

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & 3\\ 0 & 2-\lambda\end{vmatrix}=(2-\lambda)(2-\lambda)-0\cdot 3=0$$

$$\therefore\ \underline{\lambda_1=2,\ \lambda_2=2}$$

That is to say, there is *duplicity* of eigenvalues $\lambda_1=\lambda_2=2$.

2. *Eigenvectors* $\vec{v}=(x,y)$ may be calculated as

$$(A-\lambda_i I)\ \vec{v}_i=\vec{0}\ \ (i=1,2)$$

Case $\lambda_1=\lambda_2=2$:

$$(A-\lambda_1 I)\ \vec{v}_1=\left[\begin{array}{cc|c}0 & 3 & 0\\ 0 & 0 & 0\end{array}\right]\ \ (1)\leftrightarrow(2)\ \ =\left[\begin{array}{cc|c}0 & 0 & 0\\ 0 & 3 & 0\end{array}\right]\ \begin{array}{l}\Rightarrow\ \underline{x=t}\\ \Rightarrow\ 3y=0\ \ \therefore\ \ \underline{y=0}\end{array}$$

that is to say, all vectors are of the form

$$\vec{v}_1=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}t\\ 0\end{bmatrix}=t\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}$$

after the arbitrary value of parameter t is set to $t=1$ as a convenient way to derive integer coordinates. Since all eigenvectors are generated by a single eigenvector (by changing t), it is said that the eigenspace has dimension "1".

3. *Verification:*

$$A\,\vec{v}_1=\begin{bmatrix}2 & 3\\ 0 & 2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}2\cdot 1+3\cdot 0\\ 0\cdot 1+2\cdot 0\end{bmatrix}=\begin{bmatrix}2\\ 0\end{bmatrix}=2\begin{bmatrix}1\\ 0\end{bmatrix}=\lambda_1\vec{v}_1$$

8.86. Given

$$\begin{bmatrix}3 & 2 & 4\\ 2 & 0 & 2\\ 4 & 2 & 3\end{bmatrix}$$

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\begin{aligned}&\det(A-\lambda I)=\\ &=\begin{vmatrix}3-\lambda & 2 & 4\\ 2 & -\lambda & 2\\ 4 & 2 & 3-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}-\lambda & 2\\ 2 & 3-\lambda\end{vmatrix}-2\begin{vmatrix}2 & 2\\ 4 & 3-\lambda\end{vmatrix}+4\begin{vmatrix}2 & -\lambda\\ 4 & 2\end{vmatrix}\\ &=(3-\lambda)\big(-\lambda(3-\lambda)-4\big)-2\big(2(3-\lambda)-8\big)+4\big(4+4\lambda\big)\\ &=(3-\lambda)\big(\lambda^2-3\lambda-4\big)-2\big(-2\lambda-2\big)+16+16\lambda\\ &=-\lambda^3+6\lambda^2+15\lambda+8=\underline{-(\lambda-8)(x+1)^2=0}\end{aligned}$$

Three roots of this characteristic polynomial are $\lambda_1 = 8, \lambda_2 = -1,$ and $\lambda_3 = -1$, i.e., there is second-order multiplicity of $\lambda_{2,3}$ eigenvalue.

2. *Eigenvectors* $\vec{v} = (x, y)$ are calculated for each eigenvalue $\lambda_{1,2,3}$ so that

$$(A - \lambda_i I)\ \vec{v}_i = \vec{0}\ \ (i = 1, 2, 3)$$

Case $\lambda_1 = 8$:

$$(A - \lambda_1 I)\ \vec{v}_1 = \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It can be verified that determinant $|A - 8I| = 0$ indicating that at least two equations are correlated, i.e., not independent. Consequently, Cramer's rule cannot deliver the complete solution of this system of equations. By writing the extended matrix form and doing row transformations,

$$\begin{bmatrix} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{bmatrix} \begin{matrix} \\ \leftarrow (1) \div [-5] \\ \leftarrow (2) \div [2] \end{matrix} = \begin{bmatrix} 1 & -2/5 & -4/5 & 0 \\ 1 & -4 & 1 & 0 \\ 4 & 2 & -5 & 0 \end{bmatrix} \begin{matrix} \\ \leftarrow (2) - (1) \\ \leftarrow (3) - 4(2) \end{matrix}$$

$$= \begin{bmatrix} 1 & -2/5 & -4/5 & 0 \\ 0 & -18/5 & 9/5 & 0 \\ 0 & 18 & -9 & 0 \end{bmatrix} \begin{matrix} \\ \leftarrow -5/9\,(2) \\ \leftarrow (3) \div [9] \end{matrix} = \begin{bmatrix} 1 & -2/5 & -4/5 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow (3) - (2) \end{matrix}$$

$$= \begin{bmatrix} 1 & -2/5 & -4/5 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{z = t}$$

and, by substituting $z = t$ in rows (1) and (2),

$$x - \frac{2}{5}y - \frac{4}{5}z = 0 \ \Rightarrow x = \frac{2}{5}y + \frac{4}{5}z$$

$$0\,x + 2y - z = 0 \ \Rightarrow \ \underline{z = 2y}$$

$$\therefore\ x = \frac{2}{5}y + \frac{4}{5}2y = 2y \Rightarrow \underline{x = z = t}$$

so that $x = t$, $y = t/2$, and $z = t$,

$$\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t/2 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 2$ as a convenient way to derive integer coordinates.

Case $\lambda_{2,3} = -1$: (double eigenvalue)

$$\left(A-\lambda_{2,3}\,I\right)\,\vec{v}_{2,3}=\begin{bmatrix}4&2&4\\2&1&2\\4&2&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c}4&2&4&0\\2&1&2&0\\4&2&4&0\end{array}\right]\begin{array}{l}\leftarrow (1)\div[2]\\ \\ \leftarrow (3)\div[2]\end{array}=\left[\begin{array}{ccc|c}2&1&2&0\\2&1&2&0\\2&1&2&0\end{array}\right]\begin{array}{l}\\ \leftarrow (2)-(1)\\ \leftarrow (3)-(1)\end{array}=\left[\begin{array}{ccc|c}2&1&2&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

which is to say that, aside from the trivial solution $x=y=z=0$, any other linear combinations that satisfy row (1) are valid solutions, for example,

$$2x+y+2z=0\;\;\Rightarrow\;\;x=\frac{-y-2z}{2}$$

$$\therefore$$

$$\underline{\text{if } y=0:}\;\;x=-z\;\;\therefore\;\;x=t,\,z=-t\;\;\Rightarrow\;\;\vec{v}_2=\begin{bmatrix}t\\0\\-t\end{bmatrix}=t\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$\underline{\text{if } z=0:}\;\;x=-\frac{y}{2}\;\;\therefore\;\;x=t,\,y=-2t\;\;\Rightarrow\;\;\vec{v}_3=\begin{bmatrix}t\\-2t\\0\end{bmatrix}=t\begin{bmatrix}1\\-2\\0\end{bmatrix}=\begin{bmatrix}1\\-2\\0\end{bmatrix}$$

after the arbitrary value of parameter t is set to $t=1$ as a convenient way to derive integer coordinates.

3. *Verification:*

$$\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}\begin{bmatrix}2\\1\\2\end{bmatrix}=\begin{bmatrix}3\cdot2+2\cdot1+4\cdot2\\2\cdot2+0\cdot1+2\cdot2\\4\cdot2+2\cdot1+3\cdot2\end{bmatrix}=\begin{bmatrix}16\\8\\16\end{bmatrix}=8\begin{bmatrix}2\\1\\2\end{bmatrix}=\lambda_1\,\vec{v}_1$$

$$\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}3\cdot1+2\cdot0+4\cdot(-1)\\2\cdot1+0\cdot0+2\cdot(-1)\\4\cdot1+2\cdot0+3\cdot(-1)\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix}=-1\begin{bmatrix}1\\0\\-1\end{bmatrix}=\lambda_2\,\vec{v}_2$$

$$\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}\begin{bmatrix}1\\-2\\0\end{bmatrix}=\begin{bmatrix}3\cdot1+2\cdot(-2)+4\cdot0\\2\cdot1+0\cdot(-2)+2\cdot0\\4\cdot1+2\cdot(-2)+3\cdot0\end{bmatrix}=\begin{bmatrix}-1\\2\\0\end{bmatrix}=-1\begin{bmatrix}1\\-2\\0\end{bmatrix}=\lambda_3\,\vec{v}_3$$

8.87. Given

$$\begin{bmatrix}1&0&2\\-1&1&3\\0&0&2\end{bmatrix}$$

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\det(A-\lambda I)=$$

$$=\begin{vmatrix}1-\lambda & 0 & 2\\ -1 & 1-\lambda & 3\\ 0 & 0 & 2-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}1-\lambda & 3\\ 0 & 2-\lambda\end{vmatrix}-0\begin{vmatrix}\cdot & \cdot\\ \cdot & \cdot\end{vmatrix}+2\begin{vmatrix}-1 & 1-\lambda\\ 0 & 0\end{vmatrix}$$

$$=(1-\lambda)(1-\lambda)(2-\lambda)-0+2\big(-1\cdot 0-0(1-\lambda)\big)=(1-\lambda)(1-\lambda)(2-\lambda)$$

Three roots of this characteristic polynomial are $\lambda_1=2,\lambda_2=1,$ and $\lambda_3=1$, i.e., there is second-order multiplicity of $\lambda_{2,3}$ eigenvalue.

2. *Eigenvectors* $\vec{v}=(x,y)$ are calculated for each eigenvalue $\lambda_{1,2,3}$ so that

$$(A-\lambda_i I)\,\vec{v}_i=\vec{0}\ \ (i=1,2,3)$$

Case $\lambda_1=2$:

$$(A-\lambda_1 I)\,\vec{v}_1=\begin{bmatrix}-1 & 0 & 2\\ -1 & -1 & 3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

By writing the extended matrix form and doing row transformations,

$$\left[\begin{array}{ccc|c}-1 & 0 & 2 & 0\\ -1 & -1 & 3 & 0\\ 0 & 0 & 0 & 0\end{array}\right]\begin{array}{l}\leftarrow (1)\div[-1]\\ \leftarrow (2)-(1)\\ \Rightarrow\ z=t\end{array}=\left[\begin{array}{ccc|c}1 & 0 & -2 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]\begin{array}{l}\Rightarrow\ 1x+0y-2z=0\\ \Rightarrow\ 0x-1y+1z=0\\ \Rightarrow\ z=t\end{array}$$

and, by substituting $z=t$ in rows (1) and (2),

$$x-2t=0\ \Rightarrow\ x=2t$$
$$-y+t=0\ \Rightarrow\ y=t$$

so that

$$\vec{v}_1=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2t\\ t\\ t\end{bmatrix}=t\begin{bmatrix}2\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}2\\ 1\\ 1\end{bmatrix}$$

after the arbitrary value of parameter t is set to $t=1$ as a convenient way to derive integer coordinates.

Case $\lambda_2=1$: (double eigenvalue)

$$(A-\lambda_2 I)\,\vec{v}_{2,3}=\begin{bmatrix}0 & 0 & 2\\ -1 & 0 & 3\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} (1) \leftrightarrow (2) \\ \leftarrow (2) \div [-1] \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \leftarrow (1) + 3(3) \\ \leftarrow (2) - 2(3) \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \Rightarrow 1x + 0y + 0z = 0 \quad \therefore \quad \underline{x = 0} \\ \Rightarrow \underline{y = t} \\ \Rightarrow 0x + 0y + 1z = 0 \quad \therefore \quad \underline{z = 0} \end{array}$$

$$\vec{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

after the value of parameter t is conveniently set to $t = 1$. Since the third eigenvector may be generated only by changing the value of t, it is said that this eigenspace has dimension "2".

3. *Verification:*

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 0 \cdot 1 + 2 \cdot 1 \\ (-1) \cdot 2 + 1 \cdot 1 + 3 \cdot 1 \\ 0 \cdot 2 + 0 \cdot 1 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \lambda_1 \vec{v}_1$$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 0 \cdot 1 + 2 \cdot 0 \\ (-1) \cdot 0 + 1 \cdot 1 + 3 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 1 + 2 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \lambda_2 \vec{v}_2$$

8.88. Given

$$\begin{bmatrix} 1 & -2 & -1 \\ 1 & 7 & 1 \\ -2 & -4 & 3 \end{bmatrix}$$

1. *Eigenvalues* λ: the roots of characteristic polynomial are eigenvalues, as

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -2 & -1 \\ 1 & 7 - \lambda & 1 \\ -2 & -4 & 3 - \lambda \end{vmatrix}$$

$$= (1 - \lambda) \begin{vmatrix} 7 - \lambda & 1 \\ -4 & 3 - \lambda \end{vmatrix} - (-2) \begin{vmatrix} 1 & 1 \\ -2 & 3 - \lambda \end{vmatrix} + (-1) \begin{vmatrix} 1 & 7 - \lambda \\ -2 & -4 \end{vmatrix}$$

$$= (1 - \lambda)\big((7 - \lambda)(3 - \lambda) - (-4)\big) + 2\big((3 - \lambda) - (-2)\big) - \big(-4 - (-2)(7 - \lambda)\big)$$

$$= -\lambda^3 + 11\lambda^2 - 35\lambda + 25 = -(\lambda - 5)(\lambda - 5)(\lambda - 1)$$

Three roots of this characteristic polynomial are $\lambda_1 = 1, \lambda_2 = 5,$ and $\lambda_3 = 5$, i.e., there is second-order multiplicity of $\lambda_{2,3}$ eigenvalue.

2. *Eigenvectors* $\vec{v} = (x, y)$ are calculated for each eigenvalue $\lambda_{1,2,3}$ so that

$$(A - \lambda_i I)\ \vec{v}_i = \vec{0}\ \ (i = 1, 2, 3)$$

Case $\lambda_1 = 1$:

$$(A - \lambda_1 I)\ \vec{v}_1 = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 6 & 1 \\ -2 & -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form and doing row transformations,

$$\left[\begin{array}{ccc|c} 0 & -2 & -1 & 0 \\ 1 & 6 & 1 & 0 \\ -2 & -4 & 2 & 0 \end{array}\right] \begin{array}{l} (1) \leftrightarrow (2) \\ \\ \leftarrow (3) + 2(2) \end{array} = \left[\begin{array}{ccc|c} 1 & 6 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 8 & 4 & 0 \end{array}\right] \leftarrow (3) + 4(2)$$

$$= \left[\begin{array}{ccc|c} 1 & 6 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow 1x + 6y + 1z = 0 \\ \Rightarrow 0x - 2y - 1z = 0 \\ \Rightarrow \underline{z = t} \end{array}$$

and, by substituting $z = t$ in row (2) and then in row (1),

$$-2y - t = 0 \ \ \therefore \ \ \underline{y = -(1/2)\, t}$$

$$x + 6\,(-1/2)\, t + t = 0 \ \ \therefore \ \ \underline{x = 2t}$$

so that

$$\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -(1/2)\, t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -(1/2) \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 2$ as a convenient way to derive integer coordinates.

Case $\lambda_2 = 5$: (double eigenvalue)

$$(A - \lambda_2 I)\ \vec{v}_{2,3} = \begin{bmatrix} -4 & -2 & -1 \\ 1 & 2 & 1 \\ -2 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} -4 & -2 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -2 & -4 & -2 & 0 \end{array}\right] \begin{array}{l} (1) \leftrightarrow (2) \\ \\ \leftarrow (2) + 2(2) \end{array} = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -4 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow (2) + 4(1) \\ \Rightarrow \underline{z = t} \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \leftarrow (2) \div [3] \\ \Rightarrow \underline{z = t} \end{array} = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow 1x + 2y + 1z = 0 \\ \Rightarrow 0x + 2y + 1z = 0 \\ \Rightarrow \underline{z = t} \end{array}$$

and, by substituting $z = t$ in row (2) and then in row (1),

$$2y + t = 0 \quad \therefore \quad \underline{y = -(1/2)\,t}$$

$$x + 2\,(-1/2)\,t + t = 0 \quad \therefore \quad \underline{x = 0}$$

so that

$$\vec{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ (-1/2)\,t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 2$ as a convenient way to derive integer coordinates. Since the third eigenvector may be generated only by changing the value of t, it is said that this eigenspace has dimension "2".

3. *Verification:*

$$\begin{bmatrix} 1 & -2 & -1 \\ 1 & 7 & 1 \\ -2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + (-2) \cdot (-1) + (-1) \cdot 2 \\ 1 \cdot 4 + 7 \cdot (-1) + 1 \cdot 2 \\ (-2) \cdot 4 + (-4) \cdot (-1) + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \lambda_1 \vec{v}_1$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 1 & 7 & 1 \\ -2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + (-2) \cdot (-1) + (-1) \cdot 2 \\ 1 \cdot 0 + 7 \cdot (-1) + 1 \cdot 2 \\ (-2) \cdot 0 + (-4) \cdot (-1) + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \lambda_2 \vec{v}_2$$

8.8 Matrix Inversion

8.89. Given

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

1. *det*(A):

$$\Delta = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2 \neq 0 \quad \therefore \quad A^{-1} \quad \text{exists}$$

2. One possible method of matrix inversion is by Gaussian elimination, i.e., by writing the extended matrix form and doing row transformations. Starting with the "$A \mid I$" extended form, the objective is to derive "$I \mid A^{-1}$" form, as

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]}_{A \quad I} \leftarrow (2) - 3(1) \Rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \leftarrow (2) \div [-2] \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] \leftarrow (1) - 2(2) \Rightarrow \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]}_{I \quad A^{-1}}$$

Therefore,

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

3. *Verification:*

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-2) + 2 \cdot (3/2) & 1 \cdot 1 + 2 \cdot (-1/2) \\ 3 \cdot (-2) + 4 \cdot (3/2) & 3 \cdot 1 + 4 \cdot (-1/2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

8.90. Given

$$A = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix}$$

1. *det*(A):

$$\Delta = \begin{vmatrix} 4 & -2 \\ 2 & 3 \end{vmatrix} = 4 \cdot 3 - 2 \cdot (-2) = 16 \neq 0 \quad \therefore \quad A^{-1} \quad \text{exists}$$

2. One possible method of matrix inversion is by Gaussian elimination, i.e., by writing the extended matrix form and doing row transformations. Starting with the "$A \mid I$" extended form, the objective is to derive "$I \mid A^{-1}$" form, as

$$\underbrace{\left[\begin{array}{cc|cc} 4 & -2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]}_{A \quad I} \leftarrow (1) \div [4] \Rightarrow \left[\begin{array}{cc|cc} 1 & -1/2 & 1/4 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \leftarrow (2) - 2(1) \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & -1/2 & 1/4 & 0 \\ 0 & 4 & -1/2 & 1 \end{array}\right] \leftarrow (2) \div [4] \Rightarrow \left[\begin{array}{cc|cc} 1 & -1/2 & 1/4 & 0 \\ 0 & 1 & -1/8 & 1/4 \end{array}\right] \leftarrow (1) + 1/2(2) \Rightarrow$$

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 3/16 & 1/8 \\ 0 & 1 & -1/8 & 1/4 \end{array}\right]}_{I \quad A^{-1}}$$

Therefore,

$$A^{-1} = \begin{bmatrix} 3/16 & 1/8 \\ -1/8 & 1/4 \end{bmatrix}$$

3. *Verification:*

$$AA^{-1} = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3/16 & 1/8 \\ -1/8 & 1/4 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3/16 + (-2) \cdot (-1/8) & 4 \cdot 1/8 + (-2) \cdot 1/4 \\ 2 \cdot 3/16 + 3 \cdot (-1/8) & 2 \cdot 1/8 + 3 \cdot 1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or, by linear combination,

$$\begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3/16 \\ -1/8 \end{bmatrix} = 3/16 \begin{bmatrix} 4 \\ 2 \end{bmatrix} + {}^{-1}/_{8} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} (12/16 + 1/4) \\ (6/16 - 3/8) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1/8 \\ 1/4 \end{bmatrix} = 1/8 \begin{bmatrix} 4 \\ 2 \end{bmatrix} + 1/4 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} (4/8 - 1/2) \\ (2/8 + 3/4) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

8.91. Given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

1. $det(A)$:

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix} = -3 - 2(-8) + 3(-7) = -8 \neq 0 \quad \therefore \quad A^{-1} \quad \text{exists}$$

2. One possible method of matrix inversion is by Gaussian elimination, i.e., by writing the extended matrix form and doing row transformations. Starting with the "$A \mid I$" extended form, the objective is to derive "$I \mid A^{-1}$" form, as

$$\underbrace{\left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{array}\right]}_{A} \underbrace{\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}_{I} \begin{array}{l} \leftarrow (2) - 4(1) \Rightarrow \\ \leftarrow (3) - 3(1) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -8 & -4 & 1 & 0 \\ 0 & -4 & -8 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \leftarrow (2) \div [-3] \Rightarrow \\ \leftarrow (2) \div [-4] \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 8/3 & 4/3 & -1/3 & 0 \\ 0 & 1 & 2 & 3/4 & 0 & -1/4 \end{array}\right] \begin{array}{l} \\ \Rightarrow \\ \leftarrow (3) - (2) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 8/3 & 4/3 & -1/3 & 0 \\ 0 & 0 & -2/3 & -7/12 & 1/3 & -1/4 \end{array}\right] \begin{array}{l} \\ \Rightarrow \\ \leftarrow (3)[-3/2] \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 8/3 & 4/3 & -1/3 & 0 \\ 0 & 0 & 1 & 7/8 & -1/2 & 3/8 \end{array}\right] \leftarrow (2) - 8/3(3) \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 7/8 & -1/2 & 3/8 \end{array}\right] \leftarrow (1) - 2(2) \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & -2 & 2 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 7/8 & -1/2 & 3/8 \end{array}\right] \leftarrow (1) - 3(3) \Rightarrow$$

$$\underbrace{\left[\begin{array}{ccc|} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right.}_{I} \underbrace{\left.\begin{array}{ccc} 3/8 & -1/2 & 7/8 \\ -1 & 1 & -1 \\ 7/8 & -1/2 & 3/8 \end{array}\right]}_{A^{-1}}$$

Therefore,

$$A^{-1} = \begin{bmatrix} 3/8 & -1/2 & 7/8 \\ -1 & 1 & -1 \\ 7/8 & -1/2 & 3/8 \end{bmatrix}$$

3. *Verification:*

$$AA^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3/8 & -1/2 & 7/8 \\ -1 & 1 & -1 \\ 7/8 & -1/2 & 3/8 \end{bmatrix} = \cdots = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8.9 Powers of Diagonalizable Matrices

Reminder: Calculation of a matrix power is a very time-consuming process. However, one special type of matrices may be raised to powers by the process of "diagonalization". By definition,

1. A square matrix A is diagonalizable if it is similar to a diagonal matrix. This is conditioned by the existence of two extra matrices: an invertible matrix P and a diagonal matrix D. If existent, then it is possible to write the products

$$D = P^{-1}AP \quad \text{or, equivalently,}$$
$$A = P\,D\,P^{-1}$$

   where:
   (a) P is a matrix that has its inverse, i.e., $P\,P^{-1} = I$ exists
   (b) D is a diagonal matrix
2. In this form, for example,

(continued)

(continued)

$$A^5 = AAAAA = (P\,D\,P^{-1})(P\,D\,P^{-1})(P\,D\,P^{-1})(P\,D\,P^{-1})(P\,D\,P^{-1})$$
$$= PDDDDDP^{-1} = PD^5P^{-1}$$

That is to say, the problem of calculating A^5 is replaced with the problem of calculating D^5 and calculation of P^{-1} matrix, which may be easier to calculate.

8.92. Given

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^2$$

note that this matrix is already in diagonal form; therefore,

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 2^2 & 0 \\ 0 & 2^2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

8.93. Given

$$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^5$$

note that this matrix is already diagonal; therefore,

$$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^5 = \begin{bmatrix} 2^5 & 0 \\ 0 & 5^5 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 3125 \end{bmatrix}$$

8.94. Given

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

note that this matrix is not diagonal; however, it may be possible to diagonalize it.

1. Eigenvalues of A are roots of characteristic polynomial as

$$\det(A - \lambda I) = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 0 \\ -1 & 3-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda) = 0$$

   Therefore, two eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.
2. Eigenvectors are

$$\underline{\lambda_1 = 2}: \quad A - \lambda_1 I = 0 \quad \therefore \quad \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 1 & 0 \end{array}\right] \Rightarrow \underline{x = t}$$
$$\Rightarrow -x + y = 0 \quad \therefore \quad \underline{x = y = t}$$

$$\therefore \quad \vec{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

after the arbitrary $t = 1$ as convenient value, and

$$\underline{\lambda_2 = 3}: \quad A - \lambda_2 I = 0 \quad \therefore \quad \left[\begin{array}{cc|c} -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow -x + 0y = 0 \quad \therefore \quad \underline{x = 0} \\ \Rightarrow -x + 0y = 0 \quad \therefore \quad \underline{y = t} \end{array}$$

$$\therefore \quad \vec{v}_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

after the arbitrary $t = 1$ as convenient value. Note that two eigenvectors are linearly independent; in other words, matrix A is diagonalizable.

3. Diagonal matrix D consists of eigenvalues as

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

4. Matrix P and its P^{-1}, by definition, are

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

where $\underline{\det P \text{ is}}$

$$\det P = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$$

Therefore, its inverse matrix exists, and consequently, A is diagonalizable. Inverse of P may be derived by transformation "$P|I \rightarrow I|P^{-1}$" as

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \leftarrow (2) - (1) \end{array} = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \quad \therefore \quad P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

5. Now, the fifth power of diagonalizable matrix A is

$$A^5 = P\,D^5\,P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^5 & 0 \\ 0 & 3^5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 32 & 0 \\ 0 & 243 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 0 \\ 32 & 243 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ -211 & 243 \end{bmatrix}$$

which may be verified by repetitive matrix multiplications. However, note that for higher-order matrices, the above method may be more efficient than the "brute force" multiplications.

8.95. Given

$$A = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

then,

1. Eigenvalues of A are roots of characteristic polynomial as

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 2-\lambda & 0 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 4-\lambda \end{vmatrix} \\ &= (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} - 0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + 2\begin{vmatrix} -1 & 2-\lambda \\ 0 & 1 \end{vmatrix} \\ &= (2-\lambda)\big((2-\lambda)(4-\lambda)-1\big) + 2\big(-1(1)\big) = -\lambda^3 + 8\lambda^2 - 19\lambda + 12 \\ &= -(\lambda-1)(\lambda-3)(\lambda-4) = 0 \end{aligned}$$

   Three distinct eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = 4$; thus, matrix A is diagonalizable.
2. Eigenvectors are therefore
   Case $\lambda_1 = 1$:

$$(A - \lambda_1 I)\ \vec{v}_1 = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \leftarrow (2)+(1) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \leftarrow (3)-(2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \underline{z = t}$$

   and, by substituting z in rows (1) and (2),

$$\begin{aligned} 1\,x + 0\,y + 2\,z = 0 &\Rightarrow \underline{x = -2t} \\ 0\,x + 1\,y + 3\,z = 0 &\Rightarrow \underline{y = -3\,t} \end{aligned}$$

   so that

$$\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ -3\,t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

Case $\lambda_2 = 3$:

$$(A - \lambda_1 I)\ \vec{v}_1 = \begin{bmatrix} -1 & 0 & 2 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} \leftarrow (1) \div [-1] \\ \leftarrow (2) - (1) \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \leftarrow (3) + (2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \underline{z = t}$$

and, by substituting z in rows (1) and (2),

$$\begin{aligned} 1\,x + 0\,y - 2\,z = 0 &\Rightarrow \underline{x = 2t} \\ 0\,x - 1\,y - 1\,z = 0 &\Rightarrow \underline{y = -t} \end{aligned}$$

so that

$$\vec{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

Case $\lambda_3 = 4$:

$$(A - \lambda_1 I)\ \vec{v}_1 = \begin{bmatrix} -2 & 0 & 2 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow \\ \Rightarrow \\ \Rightarrow \end{array} \begin{array}{l} -2x + 0y + 2z = 0 \\ -1x + -2y + 1z = 0 \\ 0x + 1y + 0z = 0 \end{array} \begin{array}{l} \therefore \ \underline{x = z = t} \\ \therefore \ \underline{y = 0} \ \text{(because } x = z\text{)} \\ \therefore \ \underline{y = 0} \end{array}$$

so that

$$\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates. These three eigenvectors are easily verified by definition.

3. Diagonal matrix D consists of eigenvalues as

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

4. Matrix P and its P^{-1}, by definition, are

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 1 \\ -3 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

where $\underline{\det P \text{ is}}$

$$\det(P) = \begin{bmatrix} -2 & 2 & 1 \\ -3 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = -2(-1) - 2(-3) + (-3 + 1) = 6 \neq 0$$

Therefore, its inverse matrix exists, and consequently, A is diagonalizable. Inverse of P may be derived by transformation "$P|I \rightarrow I|P^{-1}$" (note: if multiple transformations are listed at the same time, then "$() \leftrightarrow ()$" is the last) as

$$\left[\begin{array}{ccc|ccc} -2 & 2 & 1 & 1 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \leftarrow (1) + 2(3) \\ \leftarrow (2) + 3(3) \\ (3) \leftrightarrow (1) \end{array} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 0 & 1 & 3 \\ 0 & 4 & 3 & 1 & 0 & 2 \end{array}\right] \begin{array}{l} \\ \leftarrow (2) \div [2] \\ \leftarrow (3) - 2(2) \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 3/2 & 0 & 1/2 & 3/2 \\ 0 & 0 & -3 & 1 & -2 & -4 \end{array}\right] \leftarrow (3) \div [-3]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 3/2 & 0 & 1/2 & 3/2 \\ 0 & 0 & 1 & -1/3 & 2/3 & 4/3 \end{array}\right] \leftarrow (2) - 3/2(3)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & -1/3 & 2/3 & 4/3 \end{array}\right] \leftarrow (1) - (3)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/3 & -2/3 & -1/3 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & -1/3 & 2/3 & 4/3 \end{array}\right] \leftarrow (1) - (2) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/6 & -1/6 & 1/6 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & -1/3 & 2/3 & 4/3 \end{array}\right]$$

$$\therefore \quad P^{-1} = \begin{bmatrix} -1/6 & -1/6 & 1/6 \\ 1/2 & -1/2 & -1/2 \\ -1/3 & 2/3 & 4/3 \end{bmatrix}$$

5. The third power of this diagonalizable matrix A is

$$\begin{aligned} A^3 = P\,D^3\,P^{-1} &= \begin{bmatrix} -2 & 2 & 1 \\ -3 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}^3 \begin{bmatrix} -1/6 & -1/6 & 1/6 \\ 1/2 & -1/2 & -1/2 \\ -1/3 & 2/3 & 4/3 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 2 & 1 \\ -3 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 64 \end{bmatrix} \begin{bmatrix} -1/6 & -1/6 & 1/6 \\ 1/2 & -1/2 & -1/2 \\ -1/3 & 2/3 & 4/3 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 54 & 64 \\ -3 & -27 & 0 \\ 1 & 27 & 64 \end{bmatrix} \begin{bmatrix} -1/6 & -1/6 & 1/6 \\ 1/2 & -1/2 & -1/2 \\ -1/3 & 2/3 & 4/3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 16 & 58 \\ -13 & 14 & 13 \\ -8 & 29 & 72 \end{bmatrix} \end{aligned}$$

which may be verified by repetitive matrix multiplications of A.

8.96. Given

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

then,

1. Eigenvalues of A are roots of characteristic polynomial as

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\big((2-\lambda)(2-\lambda) - 1\big) = -(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

Therefore, there are three distinct eigenvalues which are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. Consequently, matrix A is diagonalizable.

2. Eigenvectors are therefore

Case $\lambda_1 = 1$:

$$(A - \lambda_1 I)\ \vec{v}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \leftarrow (3) - (2) \quad = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow \quad \underline{x = 0} \\ \Rightarrow \quad \underline{y = -z = -t} \\ \Rightarrow \quad \underline{z = t} \end{array}$$

so that

$$\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

Case $\lambda_2 = 2$:

$$(A - \lambda_1 I)\ \vec{v}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow \quad \underline{x = t} \\ \Rightarrow \quad \underline{z = 0} \\ \Rightarrow \quad \underline{y = 0} \end{array}$$

so that

$$\vec{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

after the value of parameter t is conveniently set to $t = 1$.

Case $\lambda_3 = 3$:

$$(A - \lambda_1 I)\ \vec{v}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} \Rightarrow \ \underline{x = 0} \\ \Rightarrow \ \underline{y = z = t} \\ \Rightarrow \ \underline{y = z = t} \end{array}$$

so that

$$\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates. These three eigenvectors are easily verified by definition.

3. Diagonal matrix D consists of eigenvalues as

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4. Matrix P and its P^{-1}, by definition, are

$$P = \left[\, \vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \,\right] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

where $\underline{\det P \text{ is}}$

$$\det(P) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = -(-1)(1) - 1 = 2 \neq 0$$

Therefore, its inverse matrix exists, and consequently, A is diagonalizable. Inverse of P may be derived by transformation "$P|I \rightarrow I|P^{-1}$" as

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \end{array} = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow (3) + (1) \end{array}$$

$$\begin{array}{l} \\ \\ (3) \rightarrow (1) \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow (3) \div [2] \end{array} = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 \end{array}\right] \begin{array}{l} \leftarrow (1) - (3) \\ \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 \end{array}\right]$$

Therefore,

$$P^{-1} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

5. Thus, the fourth power of this diagonalizable matrix A is

$$\begin{aligned} A^4 = P\,D^4\,P^{-1} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^4 \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 16 & 0 \\ -1 & 0 & 81 \\ 1 & 0 & 81 \end{bmatrix} \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 0 & 0 \\ 0 & 41 & 40 \\ 0 & 40 & 41 \end{bmatrix} \end{aligned}$$

which may be verified by repetitive matrix multiplications of A.

8.97. Given

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

then,

Method 1:

1. Eigenvalues of A are roots of characteristic polynomial as

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \\ &= (1-\lambda)(\lambda^2 - 1) = -(\lambda - 1)(\lambda - 1)(\lambda + 1) = 0 \end{aligned}$$

Therefore, there are three eigenvalues which are $\lambda_{1,2} = 1$ (double value), and $\lambda_3 = -1$. It is necessary to verify if there are three independent eigenvectors or not.

2. Eigenvectors are therefore

Case $\lambda_3 = -1$:

$$(A - \lambda_1 I)\ \vec{v}_1 = \begin{bmatrix} 2\,0\,0 \\ 0\,1\,1 \\ 0\,1\,1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \leftarrow (1) \div [2] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} \Rightarrow \ \underline{x = 0} \\ \Rightarrow \ \underline{y = -z = -t} \\ \Rightarrow \ \underline{y = -z = -t} \end{array}$$

so that

$$\vec{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

after the arbitrary value of parameter t is set to $t = 1$ as a convenient way to derive integer coordinates.

Case $\lambda_{1,2} = 1$:

$$(A - \lambda_1 I)\ \vec{v}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By writing the extended matrix form,

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} \Rightarrow \ \underline{x = t} \\ \Rightarrow \ y = z \\ \Rightarrow \ y = z \end{array}$$

Thus, there are two possible non-trivial (i.e., $x = y = z = 0$) normalized solutions,

$$\underline{t = 1} \ : \ \Rightarrow \ \ y = z = 0$$
$$\underline{t = 0} \ : \ \Rightarrow \ \ y = z = 1$$

so that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

There are three distinct eigenvectors; therefore, A is diagonalizable (may be verified by multiplication).

3. Diagonal matrix D consists of eigenvalues as

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

4. Matrix P and its P^{-1}, by definition, are

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

where $\det P$ is

$$\det(P) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = (1)(1+1) = 2 \neq 0$$

Therefore, its inverse matrix exists, and consequently, A is diagonalizable. Inverse of P may be derived by transformation "$P|I \rightarrow I|P^{-1}$" as

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ \\ (3)-(2) \end{matrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right] \begin{matrix} \\ \\ \leftarrow (3) \div [2] \end{matrix}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right] \begin{matrix} \\ \\ \leftarrow (2)+(3) \end{matrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right]$$

Therefore,

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

5. Thus, the tenth power of diagonalizable matrix A is

$$A^{10} = P\,D^{10}\,P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{10} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Method 2:: Instead of diagonalization, it can be noted that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{10} = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{2} \right]^{5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Method 3:: or, a simple exchange of the last two rows,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{10} \quad \leftarrow (2) \leftrightarrow (3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Bibliography

[App17] W. Appel, *Mathématiques pour la physique et les physiciens*, Number 978–2–35141–339–5, 5 edn. (H&K Éditions, Harrow, 2017)

[Cou20a] MIT Open Courseware, *Multivariable Calculus*. https://ocw.mit.edu/courses/mathematics/18-02sc-multivariable-calculus-fall-2010/. Online July 2020

[Cou20b] MIT Open Courseware. *Signals and Systems*. https://ocw.mit.edu/resources/res-6-007-signals-and-systems-spring-2011. Online July 2020

[Cou20c] MIT Open Courseware. *Single Variable Calculus*. https://ocw.mit.edu/courses/mathematics/18-01sc-single-variable-calculus-fall-2010/. Online July 2020

[Dem63] B.P. Demidovič, *Collection of Problems and Exercises for Mathematical Analysis (in Russian)*, Number 517.2 D30, National Printing House for Literature in Mathematics and Physics, 5th edn. (National Printing House ,Athinaion, 1963)

[LT09] Y. Leroyer, P. Tessen, *Mathématique pour l'ingénieur: exercises et problèmes*, Number 978–2–10–052186–9, 1st edn. (Dunod, Paris, 2009)

[Mit67] D.S. Mitrinović, *Matematika I: u obliku metodičke zbirke zadataka sa rešenjima*, Number 2043, 3rd edn. (Gradjevinska Knjiga, Belgrade, 1967)

[Mit77] D.S. Mitrinovič, *Kompleksna analiza*, 4th edn. (Gradjevinska Knjiga, Belgrade, 1977)

[Mit78] D.S. Mitrinovič, *Matematika I: u obliku metodičke zbirke zadataka sa rešenjima*, Number 06–1785/1, 5rh edn. (Gradjevinska Knjiga, Belgrade, 1978)

[Mit79] D.S. Mitrinovič, *Kompleksna analiza: zbornik zadataka i problema*, vol. 3, Number 06–1431/1, 2 edn. (Naučna Knjiga, Belgrade, 1979)

[Mod64] P.S. Modenov, *Collection of Problems in Special Program (in Russian)*. National Printing House for Literature in Mathematics and Physics (National Printing House, Athinaion, 1964)

[Mü20] D. Müller, *Mathématiques*. http://www.apprendre-en-ligne.net/MADIMU2/INDEX.HTM. Online July 2020

[Sob20] R. Sobot, *Private Collection of Course Notes in Mathematics* (2020)

[Sob21] R. Sobot, *Engineering Mathematics by Example*, Number 978–3–030–79544–3, 1 edn. (Springer, Berlin, 2021)

[Spi80] M.R. Spiegel, *Analyse de Fourier et Application aux Problémes de Valeurs aux Limites*, Number France 2–7042–1019–5 , 1 edn. (McGraw–Hill, New York City, 1980)

[Ste11a] Stewart, *Analyse concepts et contextes, Vol.1: Fonctions d'une variable*, Number 978–2–8041–6306–8, 3 edn. (De Boeck Supérieur, Louvain-la-neuve, 2011)

[Ste11b] Stewart, *Analyse concepts et contextes, Vol.2: Fonctions de plusieurs variables*, Number 978–2–8041–6327–3, 3 edn. (De Boeck Supérieur, Louvain-la-neuve, 2011)

[Sym20] Symbolab, *Online calculator*. https://www.symbolab.com/. Online July 2020

[Tik20] TikZ, *Graphs generated by TikZ*. https://en.wikipedia.org/wiki/PGF/TikZ. Online July 2020

[Ven96] T.B. Vene, *Zbirka rešenih zadataka iz matematike*, vol. 2, Number 86–17–09617–9, 22 edn. (Zavod za Udžbenike i Nastavna Sredstva, Podgorica, 1996)

[Ven01] T.B. Vene, *Zbirka rešenih zadataka iz matematike*, vol. 1, Number 86–17–09031–6, 28 edn. (Zavod za Udžbenike i Nastavna Sredstva, Podgorica, 2001)

[Ven03] T.B. Vene, *Zbirka rešenih zadataka iz matematike*, vol. 3, Number 86–17–10786–3, 27 edn. (Zavod za Udžbenike i Nastavna Sredstva, Podgorica, 2003)

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023

R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41200-4

[Ven04] T.B. Vene, *Zbirka rešenih zadataka iz matematike*, vol. 4, Number 86–17–11156–9, 36 edn. (Zavod za Udžbenike i Nastavna Sredstva, Podgorica, 2004)
[Wik20a] Wikipedia, *Fourier transform*. https://en.wikipedia.org/wiki/Fourier_transform. Online July 2020
[Wik20b] Wikipedia, *Linear algebra*. https://en.wikipedia.org/wiki/Linear_algebra. Online July 2020

Index

A
Absolute numbers, 3, 8–13
Algebra, 8, 20, 36, 127, 137–170, 199–300
Amplitude, 82
Arbitrary base, 82–84, 89, 93, 94, 95

B
Bode plot, 171–197

C
Complex numbers, 137–139, 141–149, 152, 153, 163, 166, 172, 211
Complex plane, 138–139, 141–143, 147, 149–152, 164
Complex transfer function, 172

D
Determinant, 199, 202–203, 205, 206, 222–229, 234, 242–246, 250, 280

E
Eigenvalue, 199, 205–206, 247, 248, 255–285, 289–291, 293–299
Eigenvector, 199, 205–206, 247, 248, 255–285, 289–291, 293, 295, 296, 298, 299
Equations, 8, 20, 55, 76, 98, 139, 203
Euler's equation, 139, 153
Exponent, 4, 81, 89

F
Factor theorem, 20–21, 51
Fractional powers, 2–3, 5–7, 161, 162

I
Identities, 4, 7, 18, 36, 38, 40, 47, 57, 83, 89, 97, 98, 100–102, 104, 107, 110–122, 127, 137, 139, 143, 152–161, 166, 205
Inequalities, 9, 11, 12, 55–73, 77, 85, 90–96, 98, 101, 131–136

L
Linear algebra, 199–300
Linear equations, 45, 55–73, 203, 220, 221, 229, 238
Linear transformation, 199, 201–202, 217–222
Logarithm, 8, 75–96, 193–195, 197

N
Numbers, 1, 20, 57, 83, 137, 171, 199

P
Partial fraction decomposition, 21–22, 44–53
Phase, 145, 153, 172, 174–197
Piecewise linear approximation, 172, 177–197
Polynomials, 15–53, 57, 58, 85, 168, 204, 205, 256, 258, 260, 261, 263, 265, 268, 270, 271, 274, 277–284, 289, 291, 294, 297

R
Radicals, 2–3, 5–7, 161, 162

S
Special angles, 97, 110, 111, 145
System of equations, 45, 55, 60–63, 165, 204, 208, 220, 221, 228–232, 235–238, 242–245, 256, 258, 280
System of inequalities, 56, 70–73

T
Trigonometry, 97–136

V
Vector space, 204–205, 211, 217, 246–255

© The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2023
R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41200-4